REPRESENTATION THEORY

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This course will cover the representation theory of finite groups over $\mathbb{C}$. We assume the reader knows the basic properties of groups and vector spaces.

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1. Representations

1.1. Representations as matrices

Informally, a **representation** of a group is a way of writing it down as a group of matrices.

Example 1.1.1. Consider C_4 (a.k.a. $\mathbb{Z}/4$), the cyclic group of order 4:

$$C_4 = \{e, g, g^2, g^3\}$$

where $g^4 = e$ (we'll always denote the identity element of a group by e). Consider the matrices

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad M = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$M^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \qquad M^3 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Notice that $M^4 = I$. These four matrices form a subgroup of $GL_2(\mathbb{R})$, the group of all 2×2 invertible matrices with real coefficients, with the operation of matrix multiplication. This subgroup is isomorphic to C_4, the isomorphism is

$$g \mapsto M$$

and hence $g^2 \mapsto M^2, g^3 \mapsto M^3, e \mapsto I$.

Example 1.1.2. Consider the group $C_2 \times C_2$ (the *Klein-four group*) generated by g, h such that

$$g^2 = h^2 = e$$
$$gh = hg$$

Here's a representation of this group:

$$g \mapsto S = \begin{pmatrix} 1 & -2 \\ 0 & -1 \end{pmatrix}$$
$$h \mapsto T = \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix}$$

To check that this is a representation we need to check the relations:

$$S^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = T^2$$
$$ST = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = TS$$

So S and T generate a subgroup of $GL_2(\mathbb{R})$ which is isomorphic to $C_2 \times C_2$.

Now let's try and simplify by diagonalising S. The eigenvalues of S are ± 1, and the eigenvectors are

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \mapsto \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (\lambda_1 = 1)$$
$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} -1 \\ -1 \end{pmatrix} \quad (\lambda_2 = -1)$$

So if we let

$$P = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \hat{S} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Then

$$P^{-1}SP = \hat{S}$$

Now let's diagonalise T: the eigenvalues are ± 1, and the eigenvectors are

$$\begin{pmatrix}1\\0\end{pmatrix} \mapsto -\begin{pmatrix}1\\0\end{pmatrix} \quad (\lambda_1 = -1)$$
$$\begin{pmatrix}1\\1\end{pmatrix} \mapsto \begin{pmatrix}1\\1\end{pmatrix} \quad (\lambda_2 = 1)$$

Notice T and S have the same eigenvectors! Coincidence? Of course not, as we'll see later. So if

$$\hat{T} = \begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}$$

Then $P^{-1}TP = \hat{T}$.

Claim. $\hat{S}$ and $\hat{T}$ form another representation of $C_2 \times C_2$.

Proof.

$$\begin{aligned}\hat{S}^2 &= P^{-1}S^2P = P^{-1}P = I\\ \hat{T}^2 &= P^{-1}T^2P = P^{-1}P = I\\ \hat{S}\hat{T} &= P^{-1}STP = P^{-1}TSP = \hat{T}\hat{S}\end{aligned}$$

□

This new representation is easier to work with because all the matrices are diagonal, but it carries the same information as the one using S and T. We say the two representations are **equivalent**.

Can we diagonalise the representation from Example 1.1.1? The eigenvalues of M are $\pm i$, so M cannot be diagonalised over $\mathbb{R}$, but it can be diagonalized over $\mathbb{C}$. So there is some change-of-basis matrix $P \in GL_2(\mathbb{C})$ such that

$$P^{-1}MP = \hat{M} = \begin{pmatrix}i & 0\\ 0 & -i\end{pmatrix}$$

and $g \mapsto \hat{M}$ defines a representation of C_4 that is equivalent to our original $g \mapsto M$.

As the previous example shows, it's easier to work over $\mathbb{C}$ than $\mathbb{R}$.

Definition 1.1.3 (*First version*)**.** Let G be a group. A **representation** of G is a homomorphism

$$\rho : G \to GL_n(\mathbb{C})$$

for some number n.

The number n is called the **dimension** (or the **degree**) of the representation. You can define representations over any field ($\mathbb{R}, \mathbb{Q}, \mathbb{F}_p$, *etc.*) but we'll stick to $\mathbb{C}$ for almost all of this course. We'll also always assume that our groups are finite.

There's an important point to notice here: by definition, a representation ρ is a *homomorphism*, it is not just the image of that homomorphism. In particular we don't necessarily assume that ρ is an injection, *i.e.* the image of ρ doesn't have to be isomorphic to G.

If ρ is an injection, then we say that the representation is **faithful**. In our previous two examples all the representations were faithful. Here's an example of a non-faithful representation:

Example 1.1.4. Let $G = C_6 = \langle g \mid g^6 = e \rangle$. Let $n = 1$. $GL_1(\mathbb{C}) = \mathbb{C}^*$ is the group of non-zero complex numbers with the operation of multiplication. Define

$$\rho : G \to GL_1(\mathbb{C})$$

$$\rho : g \mapsto e^{\frac{2\pi i}{3}}$$

so $\rho(g^k) = e^{\frac{2\pi i k}{3}}$. We check $\rho(g)^6 = 1$, so this is a well-defined representation of C_6. But $\rho(g^3) = 1$ also, so

- the kernel of ρ is $\{e, g^3\}$.
- the image of ρ is $\left\{1, e^{\frac{2\pi i}{3}}, e^{\frac{4\pi i}{3}}\right\}$, which is isomorphic to C_3.

Example 1.1.5. Let G be any group, and n be any number. Define

$$\rho : G \to GL_n(\mathbb{C})$$

by

$$\rho : g \mapsto I_n, \quad \forall\, g \in G$$

where I_n is the $n \times n$ identity matrix. This is a representation, because:

$$\rho(g)\rho(h) = I_n I_n = I_n = \rho(gh), \quad \forall g, h \in G$$

This is called the n-dimensional **trivial representation** of G. The kernel is equal to G, and the image is the subgroup $\{I_n\} \subset GL_n$ which has only one element.

Let P be any invertible matrix. The map 'conjugate by P'

$$c_P : GL_n(\mathbb{C}) \to GL_n(\mathbb{C})$$

given by

$$c_P : M \mapsto P^{-1}MP$$

is a homomorphism. So if

$$\rho : G \to GL_n(\mathbb{C})$$

is a homomorphism, then so is $c_P \circ \rho$ (because composition of homomorphisms is a homomorphism).

Definition 1.1.6. Two representations of G

$$\rho_1 : G \to GL_n(\mathbb{C}) \qquad\qquad \rho_2 : G \to GL_n(\mathbb{C})$$

are **equivalent** if $\exists P \in GL_n(\mathbb{C})$ such that $\rho_2 = c_P \circ \rho_1$.

Equivalent representations really are 'the same' in some sense. To understand this, we have to stop thinking about matrices, and start thinking about linear maps.

1.2. Representations as linear maps

Let V be an n-dimensional vector space. The set of all invertible linear maps from V to V form a group which we call $GL(V)$.

If we pick a basis of V then every linear map corresponds to a matrix (see Corollary A.3.2), so we get an isomorphism $GL(V) \cong GL_n(\mathbb{C})$. However, this isomorphism depends on which basis we chose, and often we don't want to choose a basis at all.

Definition 1.2.1 (*Second draft of Definition 1.1.3*)**.** A **representation** of a group G is a choice of a vector space V and a homomorphism:

$$\rho : G \to GL(V)$$

So a representation of G is a pair (V, ρ) consisting of the vector space and the homomorphism. Sometimes we'll be lazy and say 'let V' be a representation' or 'let ρ be a representation' leaving the ρ or the V implicit.

If we pick a basis of V then we can identify $GL(V)$ with $GL_n(\mathbb{C})$ and we get a representation in the previous sense. If we need to distinguish between these two definitions we'll call a representation in the sense of Definition 1.1.3 a **matrix representation**.

Notice that if we set the vector space V to be $\mathbb{C}^n$ then we can use the standard basis and $GL(V)$ is exactly the same thing as $GL_n(\mathbb{C})$. So you can think of a matrix representation as a representation (in our new sense) acting on the vector space $\mathbb{C}^n$.

Lemma 1.2.2. *Let $\rho : G \to GL(V)$ be a representation of a group G. Let $\mathcal{A} = \{a_1, \ldots, a_n\}$ and $\mathcal{B} = \{b_1, \ldots, b_n\}$ be two bases for V. Then the two associated matrix representations*

$$\rho^{\mathcal{A}} : G \to GL_n(\mathbb{C})$$
$$\rho^{\mathcal{B}} : G \to GL_n(\mathbb{C})$$

are equivalent.

Proof. For each $g \in G$ we have a linear map $\rho(g) \in GL(V)$. Writing this linear map with respect to the basis $\mathcal{A}$ gives us the matrix $\rho^{\mathcal{A}}(g)$, and writing it with respect to the basis $\mathcal{B}$ gives us the matrix $\rho^{\mathcal{B}}(g)$. Then by Corollary A.3.2 we have

$$\rho^{\mathcal{B}}(g) = P^{-1}\rho^{\mathcal{A}}(g)P$$

where P is the change-of-basis matrix between $\mathcal{A}$ and $\mathcal{B}$. This is true for all $g \in G$, so

$$\rho^{\mathcal{B}} = c_P \circ \rho^{\mathcal{A}}$$

□

Conversely, suppose ρ_1 and ρ_2 are equivalent matrix representations of G, and let P be the matrix such that $\rho_2 = c_P \circ \rho_1$. Set V to be the vector space $\mathbb{C}^n$, so we can think of ρ_1 as a representation:

$$\rho_1 : G \to GL(V)$$

Now let $\mathcal{B} \subset \mathbb{C}^n$ be the basis consisting of the columns of the matrix P, so P is the change-of-basis matrix between the standard basis and $\mathcal{B}$ (see Section A.3). For each group element g, if we write down the linear map $\rho_1(g)$ using the basis $\mathcal{B}$ then we get the matrix

$$P^{-1}\rho_1(g)P = \rho_2(g)$$

So we can view ρ_2 as the matrix representation that we get when we take ρ_1 and write it down using the basis $\mathcal{B}$.

Moral: Two matrix representations are equivalent if and only if they describe the same representation in different bases.

1.3. Constructing representations

Recall that the symmetric group S_n is the group of all permutations of a set of n symbols. There is an n-dimensional representation of S_n called the **permutation representation**. Here's how to construct it.

Let V be an n-dimensional vector space with a basis $\{b_1, \dots, b_n\}$. Every element $g \in S_n$ is a permutation of the set $\{1, \dots, n\}$ (or if you prefer, it's a permutation of the set $\{b_1, \dots, b_n\}$). Define a linear map

$$\rho(g) : V \to V$$

by setting

$$\rho(g) : b_k \mapsto b_{g(k)}$$

and extending this to a linear map. This linear map $\rho(g)$ is invertible, it has inverse $\rho(g^{-1})$, so we have defined a function:

$$\rho : G \to GL(V)$$

If we take two group elements g, h then

$$\rho(g) \circ \rho(h) : b_k \mapsto b_{gh(k)}$$

so we must have that

$$\rho(g) \circ \rho(h) = \rho(gh)$$

since these two linear maps agree on a basis. So ρ is indeed a representation of G.

Example 1.3.1. Consider the group:

$$S_3 = \{(1),\ (123),\ (132),\ (12),\ (23),\ (13)\}$$

The permutation representation of G is the homomorphism $\rho : S_3 \to GL_3(\mathbb{C})$ with:

$$\rho : (1) \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \rho : (12) \mapsto \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\rho : (123) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \qquad \rho : (23) \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\rho : (132) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \qquad \rho : (13) \mapsto \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

Aside: this construction works over any field.

Now suppose we have a group G which is a subgroup of S_n. We can get an n-dimensional representation of G by taking the permutation representation of S_n, and looking at what it does to the elements of $G \subset S_n$. This will give a representation of G because of the following trivial lemma:

Lemma 1.3.2. *Let G be a group and let $H \subset G$ be a subgroup. Let $\rho : G \to GL(V)$ be a representation of G. Then the restriction of ρ to H*

$$\rho|_H : H \to GL(V)$$

is a representation of H.

Proof. Obvious. □

Example 1.3.3. Let $G = \{(1), (12)\} \subset S_3$. This is a subgroup, and it's isomorphic to the cyclic group $C_2 = \langle g \,|\, g^2 = e \rangle$. Restricting the permutation representation of S_3 to G gives us a 3-dimensional representation:

$$\begin{aligned} \rho :\ C_2 &\to GL_3(\mathbb{C}) \\ g &\mapsto \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{aligned}$$

But of course there are other ways to realize C_2 as a subgroup of some S_n. For example consider $G' = \{(1), (12)(34)\} \subset S_4$. Then G' is also isomorphic to C_2, and restricting the permutation representation of S_4 to G' produces a 4-dimensional representation:

$$\begin{aligned} \rho :\ C_2 &\to GL_4(\mathbb{C}) \\ g &\mapsto \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \end{aligned}$$

So this construction can produce many different representations of a given group G.

However, for any group G of size n, there is one special ("canonical") way to realise G as a subgroup of the symmetric group S_n. You may remember it:

Theorem 1.3.4 (Cayley's Theorem)**.** *Any group of size n is a subgroup of the symmetric group S_n.*

Proof. Label the elements of G as $g_1, ..., g_n$. For each $g \in G$, left multiplication by g defines a function:

$$\begin{aligned} \mathcal{L}_g &: G \to G \\ \mathcal{L}_g &: g_i \mapsto gg_i = g_j, \quad \text{for some } j \end{aligned}$$

And $\mathcal{L}_g$ is a bijection since $\mathcal{L}_{g^{-1}}$ is the inverse function. So $\mathcal{L}_g$ corresponds to some permutation of the set $\{1, ..., n\}$, and composing elements in G corresponds to composing permutations. □

So for any group of size n we automatically get an n-dimensional representation of G. This is called the **regular representation**, we denote it:

$$\rho_{reg} : G \to GL_n(\mathbb{C})$$

It's very important! We will see why later.

Example 1.3.5. Let $G = C_2 \times C_2 = \langle g, h \,|\, gh = hg \rangle$. Left multiplication by g and h give permutations

$$\begin{aligned} \mathcal{L}_g : G &\to G \\ e &\mapsto g \\ g &\mapsto e \\ h &\mapsto gh \\ gh &\mapsto h \end{aligned} \qquad\qquad \begin{aligned} \mathcal{L}_h : G &\to G \\ e &\mapsto h \\ g &\mapsto gh \\ h &\mapsto e \\ gh &\mapsto g \end{aligned}$$

If we number the elements of G as $g_1 = e,\ g_2 = g,\ g_3 = h,\ g_4 = gh$ then the regular representation of G is the homomorphism

$$\rho_{reg} : G \to GL_4(\mathbb{C})$$

with:

$$\rho_{reg}(g) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad \text{and} \quad \rho_{reg}(h) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

Notice that to write down the regular represention we had to choose an ordering of our group elements (*i.e.* we had to label them as $g_1, ..., g_n$) and if we choose a different ordering then we will get a different set of matrices for ρ_{reg}. Fortunately different choices will give equivalent matrix representations, because re-ordering the group elements corresponds to re-ordering the standard basis of $\mathbb{C}^n$.

A better way to handle this is to avoid choosing an ordering at all. Let's declare V_{reg} to be an n-dimensional vector space which has a basis

$$\{b_g,\ g \in G\}$$

indexed by the elements of G. Then any $h \in G$ defines a permutation $\mathcal{L}_h : G \to G$, which we can extend to a linear map:

$$\begin{aligned} \rho_{reg}(h) : V_{reg} &\to V_{reg} \\ b_g &\mapsto b_{hg} \end{aligned}$$

This is a more elegant way to define the regular representation, and we will need to use this description in later sections.

Permutation representations are useful but they're not the only way to construct representations.

Example 1.3.6. Let $G = S_n$ for some n. Recall that there is a homomorphism from S_n to the group $\{1, -1\}$ given by

$$sgn(\sigma) = \begin{cases} 1 & \text{if } \sigma \text{ is an even permutation} \\ -1 & \text{if } \sigma \text{ is an odd permutation} \end{cases}$$

This gives a 1-dimensional representation of S_n:

$$\begin{aligned} \rho : S_n &\to GL_1(\mathbb{C}) \\ \sigma &\mapsto sgn(\sigma) \end{aligned}$$

It's called the **sign representation** of S_n.

For some groups we can construct representations using geometry.

Example 1.3.7. D_4 is the symmetry group of a square. It has size 8, and consists of 4 reflections and 4 rotations. Draw a square in the plane with vertices at the four points $(\pm 1, \pm 1)$. Then the elements of D_4 naturally become linear maps acting on a 2-dimensional vector space. Using the standard basis, we get the matrices:

- rotate by $\frac{\pi}{2}$: $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$
- rotate by π: $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$

- rotate by $\frac{3\pi}{2}$: $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$
- reflect in x-axis: $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
- reflect in y-axis: $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$
- reflect in $y = x$: $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
- reflect in $y = -x$: $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$

Together with I_2, these matrices give a 2-dimensional representation of D_4.

Finally, if we have a homomorphism of groups

$$f : H \to G$$

then we can use representations of G to construct representations of H.

Lemma 1.3.8. *Let G and H be two groups, let $f : H \to G$ be a homomorphism, and let (V, ρ) be a representation of G. Then*

$$\rho \circ f : H \to GL(V)$$

is a representation of H.

Proof. A composition of homomorphisms is a homomorphism. □

If H is a subgroup of G then the inclusion map $i : H \hookrightarrow G$ is a homomorphism. So Lemma 1.3.2 is just a special case of this lemma.

Example 1.3.9. Let $H = C_6 = \langle h \,|\, h^6 = e \rangle$ and let $G = C_3 = \langle g \,|\, g^3 = e \rangle$. Let $f : H \to G$ be the homomorphism defined by $f : h \mapsto g$. There's a faithful 1-dimensional representation of C_3 defined by

$$\rho : G \to GL_1(\mathbb{C})$$
$$\rho : g \mapsto e^{\frac{2\pi i}{3}}$$

Then $\rho \circ f$ is the non-faithful representation of C_6 that we looked at in Example 1.1.4.

Example 1.3.10. Let $G = D_6$, the symmetry group of a regular hexagon. Now consider the three diagonal lines in the hexagon, joining opposite corners. Any symmetry of the hexagon will induce some permutation of the three diagonals. This gives us a homomorphism:

$$f : D_6 \to S_3$$

If we compose f with the permutation representation of S_3 we get a 3-dimensional representation of D_6.

1.4. Subrepresentations and G-linear maps

Example 1.4.1. Let $G = C_2 \times C_2 = \langle g, h \,|\, g^2 = h^2 = e,\, gh = hg \rangle$ and let (V, ρ) be the representation of G from Example 1.1.2. So $V = \mathbb{C}^2$ and:

$$\rho(g) = S = \begin{pmatrix} 1 & -2 \\ 0 & -1 \end{pmatrix} \qquad \rho(h) = T = \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix}$$

Consider the 1-dimensional subspace:

$$U = \left\{ \begin{pmatrix} \lambda \\ \lambda \end{pmatrix},\ \lambda \in \mathbb{C} \right\} \subset V$$

spanned by the vector $(1,1)^T$. This vector is an eigenvector of both S and T, so it is also an eigenvector of the matrix $\rho(gh) = ST$. This means that the subspace U is preserved by all the linear maps appearing in this representation, *i.e.* if we take any vector $\mathbf{x} \in U$ then we have:

$$S\mathbf{x} \in U \quad \text{and} \quad T\mathbf{x} \in U \quad \text{and} \quad ST\mathbf{x} \in U$$

So we can view S and T and ST as linear maps acting on the subspace U. This means that U gives us a 1-dimensional representation of G. In fact, looking at the eigenvalues, we have:

$$\begin{aligned} S : U &\to U \\ \mathbf{x} &\mapsto -\mathbf{x} \end{aligned} \qquad\qquad \begin{aligned} T : U &\to U \\ \mathbf{x} &\mapsto \mathbf{x} \end{aligned}$$

So (U, ρ) is the 1-dimensional representation of G with:

$$g \mapsto -1 \qquad \text{and} \qquad h \mapsto 1$$

Definition 1.4.2. A **subrepresentation** of a representation $\rho : G \to GL(V)$ is a linear subspace

$$U \subset V$$

such that for all $g \in G$ and all $x \in U$ we have:

$$\rho(g)(x) \in U$$

This says that $U \subset V$ is preserved by every linear map $\rho(g)$ appearing in the representation. So for every $g \in G$ we get a linear map

$$\rho(g)|_U : U \to U$$

and it follows immediately that this gives a representation of G on the vector space U.

The easiest kind of subrepresentations are the 1-dimensional ones. Take any representation (V, ρ). If $U \subset V$ is a 1-dimensional subspace then:

- U is the span of a single vector $\mathbf{x} \in V$.
- If we have a linear map $T : V \to V$ then T preserves U iff $T\mathbf{x} = \lambda\mathbf{x}$ for some $\lambda \in \mathbb{C}$, *i.e.* iff $\mathbf{x}$ is an eigenvector of T.

So a 1-dimensional subrepresentation of (V, ρ) is the same thing as a *simultaneous eigenvector* of all the $\rho(g)$'s, *i.e.* a vector $\mathbf{x} \in V$ with:

$$\rho(g)\colon \mathbf{x} \mapsto \lambda_g \mathbf{x}, \quad \forall g \in G$$

The eigenvalues λ_g must define a homomorphism:

$$\begin{aligned} \rho' : G &\to GL_1(\mathbb{C}) \\ g &\mapsto \lambda_g \end{aligned}$$

This is our 1-dimensional representation of G on the subspace $U = \langle \mathbf{x} \rangle$. We saw one example of this in Example 1.4.1. Here is another one:

Example 1.4.3. Set $V = \mathbb{C}^3$ and let $\rho : S_3 \to GL_3(\mathbb{C})$ be the permutation representation, as in Example 1.3.1. Consider the vector $\mathbf{x} = (1,1,1)^\top$. If we take any permutation $\sigma \in S_3$ then

$$\rho(g) : \mathbf{x} \mapsto \mathbf{x}$$

i.e. $\mathbf{x}$ is an eigenvector of $\rho(g)$ with eigenvalue 1. So the subspace

$$U = \langle \mathbf{x} \rangle \subset V$$

is a 1-dimensional subrepresentation. If we think about U on its own we have a 1-dimensional representation (U, ρ') and this is the trivial 1-dimensional representation of S_3.

But not every subrepresentation is 1-dimensional.

Example 1.4.4. Again let $V = \mathbb{C}^3$ and $\rho : S_3 \to GL_3(\mathbb{C})$ be the permutation representation. Consider the 2-dimensional subspace:

$$W = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad x + y + z = 0 \right\} \subset \mathbb{C}^3$$

For any $g \in S_3$ the matrix $\rho(g)$ preserves the subspace W, since $\rho(g)$ just permutes the values x, y, z. Hence W is a subrepresentation of V. And, by restricting each $\rho(g)$ to W, we get a 2-dimensional representation:

$$\rho_U : S_3 \to GL(W)$$

If we want to understand this more explicitly we can pick a basis for W, for example we could choose:

$$\mathbf{x}_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \quad \mathbf{x}_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

Then we compute that

$$\rho\big((123)\big) : \mathbf{x}_1 \mapsto \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = \mathbf{x}_2$$

$$\mathbf{x}_2 \mapsto \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = -\mathbf{x}_1 - \mathbf{x}_2$$

So using this basis the linear map $\rho_W\big((123)\big)$ is represented by the matrix $\begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$. We can do a similar calculation for the other elements of S_3, for example:

$$\rho_W\big((12)\big) \mapsto \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}$$

This gives us a 2-dimensional matrix representation of S_3.

We now want to start thinking about functions *between* representations.

Definition 1.4.5. Let (V, ρ_V) and (W, ρ_W) be two representations of G on vector spaces V and W. A **G-linear map** between them is a linear map $f : V \to W$ such that

$$f \circ \rho_V(g) = \rho_W(g) \circ f \qquad \forall\, g \in G$$

i.e. both ways round the square

$$\begin{array}{ccc} V & \xrightarrow{\rho_V(g)} & V \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ W & \xrightarrow{\rho_W(g)} & W \end{array}$$

give the same answer ('the square commutes').

So a G-linear map is a special kind of linear map that respects the group actions. For any linear map, we have

$$f(\lambda x) = \lambda f(x)$$

for all $\lambda \in \mathbb{C}$ and $x \in V$, *i.e.* we can pull scalars through f. For G-linear maps we also have

$$f\big(\rho_V(g)(x)\big) = \rho_W(g)\big(f(x)\big)$$

for all $g \in G$, *i.e.* we can also pull group elements through f.

Claim 1.4.6. *Suppose we have two representations (V, ρ_V) and (W, ρ_W) of G and*

$$f : V \to W$$

is a G-linear map between them. Then $\mathrm{Ker}(f)$ is a subrepresentation of V and $\mathrm{Im}(f)$ is a subrepresentation of W.

Example 1.4.7. Let $G = S_3$. Let $V = \mathbb{C}^3$ and ρ_V be the permutation representation. Also let $U = \mathbb{C}$, and ρ_U be the 1-dimensional trivial representation of G. Define a linear map $f : V \to U$ by:

$$f : \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto x + y + z$$

Then f is also G-linear because for any permutation $g \in S_3$ we have:

$$f\big(\rho_V(g)(\mathbf{x})\big) = f(\mathbf{x}) = \rho_U(g)\big(f(\mathbf{x})\big)$$

The kernel of f is the 2-dimensional subrepresentation $W \subset V$ from Example 1.4.4. The image of f is the whole of U.

Example 1.4.8. Let $G = C_2 \times C_2$. Let $V = \mathbb{C}^2$ and ρ_V be the representation defined by the matrices S and T from Example 1.1.2. Also let $W = \mathbb{C}^2$, but let ρ_W be the second matrix representation

$$\rho_W(g) = \hat{S} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \rho_W(h) = \hat{T} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

from that example. Now consider the linear map $f : V \to W$ given by the matrix:

$$F = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$$

We compute that:

$$FS = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} = \hat{S}F \qquad \text{and} \qquad FT = \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix} = \hat{T}F$$

This says that $f \circ \rho_V(g) = \rho_W(g) \circ f$ and $f \circ \rho_V(h) = \rho_W(h) \circ f$, so the map f is G-linear. It's not necessary to check the corresponding equation for the final group element gh, that will follow automatically.

The kernel of f is the 1-dimensional subrepresentation $U = \langle \left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right) \rangle$ from Example 1.4.1. The image of f is the subspace

$$\left\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\rangle \subset \mathbb{C}^2$$

which is a 1-dimensional subrepresentation of W.

In fact we saw another G-linear map between these representations already in Example 1.1.2. Recall the change-of-basis matrix

$$P = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

which satisfies the equations $SP = \hat{S}P$ and $TP = \hat{T}P$, *i.e.* it shows that the two representations are equivalent. This matrix is a G-linear map from W to V. The inverse matrix

$$P^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

is a G-linear map from V to W.

Definition 1.4.9. An **isomorphism** between two representations (V, ρ_V) and (W, ρ_W) of G is a G-linear map $f : V \to W$ which is also an isomorphism of vector spaces.

Recall that if a linear map $f : V \to W$ is an isomorphism (*i.e.* a bijection) then the inverse function $f^{-1} : W \to V$ is automatically linear. Similarly:

Claim 1.4.10. *If $f : V \to W$ is an isomorphism of representations then f^{-1} is also a G-linear map.*

Isomorphism of representations is really the same thing as equivalence.

Proposition 1.4.11. *Let (V, ρ_W) and (W, ρ_W) be two representations of a group G, with* $\dim V = \dim W = n$*. Let $\mathcal{A} = \{a_1, \ldots, a_n\}$ be a basis for V, let $\mathcal{B} = \{b_1, \ldots, b_n\}$ be a basis for W, and let*

$$\rho_V^{\mathcal{A}} : G \to GL_n(\mathbb{C})$$
$$\rho_W^{\mathcal{B}} : G \to GL_n(\mathbb{C})$$

be the matrix representations obtained by writing ρ_V and ρ_W in these bases. Then ρ_V and ρ_W are isomorphic if and only if $\rho_V^{\mathcal{A}}$ and $\rho_W^{\mathcal{B}}$ are equivalent.

Proof. ($\Rightarrow$) Let $f : V \to W$ be a G-linear isomorphism. Then

$$f\mathcal{A} = \{f(a_1), \ldots, f(a_n)\} \subset W$$

is a second basis for W. Let $\rho_W^{f\mathcal{A}}$ be the matrix representation obtained by writing down ρ_W in this new basis. Pick $g \in G$ and let $\rho_V^{\mathcal{A}}(g) = M$, *i.e.*

$$\rho_V(g)(a_k) = \sum_{i=1}^{n} M_{ik} a_i$$

(see Section A.2). Then by the G-linearity of f,

$$\begin{aligned} \rho_W(g)(f(a_k)) &= f(\rho_V(g)(a_k)) \\ &= \sum_{i=1}^{n} M_{ik} f(a_i) \end{aligned}$$

So the matrix describing $\rho_W(g)$ in the basis $f\mathcal{A}$ is the matrix M, *i.e.* it is the same as the matrix describing $\rho_V(g)$ in the basis $\mathcal{A}$. This is true for all $g \in G$, so the two matrix representations $\rho_V^{\mathcal{A}}$ and $\rho_W^{f\mathcal{A}}$ are identical. And by Lemma 1.2.2 we know that $\rho_W^{f\mathcal{A}}$ is equivalent to $\rho_W^{\mathcal{B}}$.

($\Leftarrow$) Let P be the matrix such that

$$\rho_W^{\mathcal{B}} = c_P \circ \rho_V^{\mathcal{A}}$$

Let

$$f : V \to W$$

be the linear map represented by the matrix P^{-1} with respect to the bases $\mathcal{A}$ and $\mathcal{B}$. Then f is an isomorphism of vector spaces because P^{-1} is an invertible matrix. We need to show that f is also G-linear, *i.e.* that

$$f \circ \rho_V(g) = \rho_W(g) \circ f, \qquad \forall g \in G$$

Using our given bases we can write each of these linear maps as matrices, then this equation becomes

$$P^{-1}\rho_V^{\mathcal{A}}(g) = \rho_W^{\mathcal{B}}(g)P^{-1}, \qquad \forall g \in G$$

or equivalently

$$\rho_W^{\mathcal{B}}(g) = P^{-1}\rho_V^{\mathcal{A}}(g)P, \qquad \forall g \in G$$

and this is true by the definition of P. $\square$

1.5. Maschke's theorem

Let U and W be two vector spaces. Recall the definition of the **direct sum**:

$$V = U \oplus W$$

It's the vector space of all pairs (x, y) such that $x \in U$ and $y \in W$. Its dimension is $\dim U + \dim W$.

U sits inside V as the subspace consisting of all vectors of the form $(x, 0), x \in U$. Similarly W is the subspace $\{(0, y), y \in W\}$. The intersection of these two subspaces is just the zero vector $(0, 0)$. You should recall:

Lemma 1.5.1. *Suppose V is a vector space and $U, W \subset V$ are two subspaces such that:*

(i) $U \cap W = \{0\}$
(ii) $\dim U + \dim W = \dim V$

Then we have an isomorphism of vector spaces:

$$\begin{aligned} U \oplus W &\to V \\ (x, y) &\mapsto x + y \end{aligned}$$

If we have just one subspace $U \subset V$ we can look for another subspace W that satisfies the conditions of the lemma, this is called a *complementary subspace* to U. Complementary subspaces always exist, but it's important to remember that they are not unique.

Example 1.5.2. Let $V = \mathbb{C}^2$ and let $U \subset V$ be the span of the vector $\binom{1}{0}$, *i.e.* the x-axis. Then the span of $\binom{0}{1}$ is a complementary subspace to U. But there are many other choices, for example the span of $\binom{1}{1}$ works, and so does the span of $\binom{2}{1}$. In fact if we take any vector $\mathbf{x} \in \mathbb{C}^2$ which is linearly independent of $\binom{1}{0}$ then

$$W = \langle \mathbf{x} \rangle \ \subset V$$

is a complementary subspace to U.

Now we return to representation theory.

Suppose G is a group, and we have representations (U, ρ_U) and (W, ρ_W) of G. Then there is a natural representation of G on the vector space $V = U \oplus W$ given by 'direct-summing' ρ_U and ρ_W. The definition is

$$\begin{aligned} \rho_{U\oplus W} \colon\ & G \to GL(U \oplus W) \\ \rho_{U\oplus W}(g) \colon\ & (x, y) \mapsto \big(\rho_U(g)(x),\ \rho_W(g)(y)\big) \end{aligned}$$

Claim 1.5.3. *For each g, $\rho_{U\oplus W}(g)$ is linear and invertible, and $\rho_{U\oplus W}$ is indeed a homomorphism from G to $GL(U \oplus W)$.*

Pick a basis $\{a_1, \ldots, a_n\}$ for U, and $\{b_1, \ldots, b_m\}$ for W. Suppose that in these bases $\rho_U(g)$ is the matrix M and $\rho_W(g)$ is the matrix N. The set

$$\{(a_1, 0), \ldots, (a_n, 0), (0, b_1), \ldots, (0, b_m)\}$$

is a basis for $U \oplus W$, and in this basis the linear map $\rho_{U\oplus W}(g)$ is given by the matrix

$$\begin{pmatrix} M & 0 \\ 0 & N \end{pmatrix} \qquad \in \mathrm{Mat}_{(n+m)\times(n+m)}(\mathbb{C})$$

A matrix like this is called **block-diagonal**.

We have our two subspaces $U \subset V$ and $W \subset V$, and it is clear from the definition that these are subrepresentations.

Lemma 1.5.4. *Let (V, ρ) be a representation of G, and let $U \subset V$ and $W \subset V$ be subrepresentations such that*

(i) $U \cap W = \{0\}$
(ii) $\dim U + \dim W = \dim V$

Then V and $U \oplus W$ are isomorphic representations.

Proof. From Lemma 1.5.1 we have an isomorphism of vector spaces

$$\begin{aligned} f : U \oplus W &\to V \\ (x, y) &\mapsto x + y \end{aligned}$$

so we only need to check that f is G-linear. The representation U is defined by

$$\rho_U(g) = \rho(g)|_U : U \to U$$

for each $g \in G$, and similarly for ρ_W. So for any $(x, y) \in U \oplus W$ we have the square

$$\begin{tikzcd} (x,y) \arrow[r, mapsto, "\rho_{U\oplus W}(g)"] \arrow[d, mapsto, "f"] & (\rho_U(g)(x),\ \rho_W(g)(y)) \arrow[d, mapsto, "f"] \\ x + y \arrow[r, mapsto, "\rho_V(g)"] & \rho(g)(x+y) = \rho_U(g)(x) + \rho_W(g)(y) \end{tikzcd}$$

and this commutes because $\rho(g)$ is linear. So f is an isomorphism of representations. ☐

Now suppose we have a representation (V, ρ) of G and we have found one subrepresentation $U \subset V$. A second subrepresentation $W \subset V$ which satisfies the conditions of the lemma is called a **complementary subrepresentation** to U. If we can find a complementary subrepresentation then we can split V up as a direct sum $U \oplus W$, and make all of our matrices block-diagonal.

This sounds good, but is it always possible? We know we can always find a complementary *subspace* to U, in fact there are infinitely-many complementary subspaces. But it's not clear if any of them will be subrepresentations, *i.e.* preserved by all the linear maps $\rho(g)$.

Example 1.5.5. Let $G = C_2 = \langle g \,|\, g^2 = e\rangle$. Let (V, ρ) be the regular representation of G, so $V = \mathbb{C}^2$ and:

$$\rho(g) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

(this is just the permutation representation of S_2). A 1-dimensional subrepresentation must be the span of an eigenvector of $\rho(g)$, so for example

$$U = \left\langle \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\rangle \subset V$$

is a subrepresentation. This U is isomorphic to the 1-dimensional trivial representation of G, because the eigenvalue is 1, so $\rho(g)|_U$ is the identity map on U.

There is exactly one possible complementary subrepresentation to U, it's the span of the other eigenvector:

$$W = \left\langle \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\rangle \subset V$$

This is another 1-dimensional representation of G. It's isomorphic to the sign representation of S_2 (Example 1.3.6) because

$$\rho(g)|_W = -\mathbf{1}_W$$

i.e. the eigenvalue is -1. So V is isomorphic to the direct sum of these two 1-dimensional representations. And if we use the basis given by the eigenvectors then $\rho(g)$ becomes a diagonal matrix:

$$\rho(g) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Example 1.5.6. Let $G = S_3$ and let (V, ρ) be the permutation representation. In Example 1.4.3 we found a 1-dimensional subpresentation

$$U = \left\{ \begin{pmatrix} x \\ x \\ x \end{pmatrix}, \; x \in \mathbb{C} \right\} \subset V$$

spanned by the vector $(1, ..., 1)^\top$. A complementary subpresentation to U must have dimension 2, and in Example 1.4.4 we found a 2-dimensional subrepresentation:

$$W = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \; x + y + z = 0 \right\} \subset V$$

It's easy to see that W is complementary to U, so V is isomorphic to $U \oplus W$ as a representation.

If we take the basis $\mathbf{x}_1, \mathbf{x_2} \in U$ from Example 1.4.4 and add in $\mathbf{x}_0 = (1, 1, 1)^\top$ then we have a basis for $V = U \oplus W$, and in this basis all matrices $\rho(g)$ will be block diagonal. For example:

$$\rho\big((123)\big) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix} \qquad \text{and} \qquad \rho\big((12)\big) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

Example 1.5.7. We can generalize the previous two examples to $G = S_n$, for any n. Take the permutation representation $V = \mathbb{C}^n$, and set:

$$U = \langle (1, ..., 1)^T \rangle \qquad \text{and} \qquad W = \{\mathbf{x} \in \mathbb{C}^n,\ x_1 + ... + x_n = 0\}$$

Then U is a 1-dimensional subrepresentation, isomorphic to the trivial representation, and W is a subrepresentation of dimension $n-1$. They are complementary so $V = U \oplus W$.

In the examples above we were able to find a complementary subrepresentation to our initial $U \subset V$. As it turns out, this is always possible.

Theorem 1.5.8 (Maschke's Theorem)**.** *Let (V, ρ) be a representation of G and let $U \subset V$ be a subrepresentation. Then there exists a complementary subrepresentation $W \subset V$ to U.*

This is the single most important theorem in this course. To prove it we need a little bit more linear algebra.

Lemma 1.5.9. *Let V be a vector space and let $U \subset V$ be a subspace. Suppose we have a linear map $f : V \to V$ such that $\mathrm{Im}(f) = U$ and:*

$$f^2 = f$$

Then $\mathrm{Ker}(f) \subset V$ is a complementary subspace to U, i.e.

$$V = U \oplus \mathrm{Ker}(f)$$

Proof. Suppose $x \in U$. Then by assumption $x = f(y)$ for some $y \in V$, hence:

$$f(x) = f^2(y) = f(y) = x$$

So f acts as the identity on the subspace U. In particular if $x \in \mathrm{Ker}(f) \cap U$ then $f(x) = x = 0$, so $\mathrm{Ker}(f) \cap U = \{0\}$. And by the Rank-Nullity Theorem we have:

$$\dim \mathrm{Ker}(f) + \dim U = \dim V$$

□

A linear map like this is called a **projection**. For example, suppose that $V = U \oplus W$, and let π_U be the linear map

$$\begin{aligned} \pi_U : V &\to V \\ (x, y) &\mapsto (x, 0) \end{aligned}$$

Then π_U is a projection and $\mathrm{Ker}(\pi_U) = W$. The above lemma says that every projection looks like this.

Corollary 1.5.10. *Let (V, ρ) be a representation of G and $U \subset V$ a subrepresentation. Suppose we have a G-linear projection*

$$f : V \to V$$

with $\mathrm{Im}(f) = U$. Then $\mathrm{Ker}(f)$ is a complementary subrepresentation to U.

Proof. This is immediate from the previous lemma and Claim 1.4.6. □

We also need the following trick, which says that projections onto a fixed subspace $U \subset V$ can be averaged.

Lemma 1.5.11. *Let V be a vector space and let $U \subset V$ be a subspace. Suppose we have linear maps*

$$f_i : V \to V$$

for $i = 1, .., n$ such that each f_i is a projection with $\mathrm{Im}(f_i) = U$. Then the linear map

$$f = \frac{1}{n}(f_1 + ... + f_n)$$

is also a projection with $\mathrm{Im}(f) = U$.

Proof. For any $x \in V$ we have $f_i(x) \in U$ for each i, hence $f(x) \in U$ since U is a subspace. So $\mathrm{Im}(f) \subset U$. And if x lies in U then we have $f_i(x) = x$ for all i, so:

$$f(x) = \frac{1}{n}\big(f_1(x) + ... + f_n(x)\big) = \frac{1}{n}(nx) = x$$

Hence $\mathrm{Im}(f)$ is exactly U, and also $f^2 = f$. □

Now we can prove Maschke's Theorem.

Proof of Theorem 1.5.8. By Corollary 1.5.10, it's enough to find a G-linear projection on V with image W. Start by picking any complementary subspace

$$\tilde{W} \subset V$$

to U (this $\tilde{W}$ is just a complementary subspace, not a complementary subrepresentation, so we know we can always find one). Let

$$\tilde{f} : V \to V$$

be the corresponding projection with kernel $\tilde{W}$ and image U. There is no reason why $\tilde{f}$ should be G-linear. However, we can do a clever modification.
Pick some group element $g \in G$, and consider the map:

$$\rho(g) \circ \tilde{f} \circ \rho(g^{-1}) : V \to V$$

We claim this again a projection with image U. To see this, take any $x \in V$, and note that

$$\tilde{f}(\rho(g^{-1})(x)) \in U$$

because the image of $\tilde{f}$ is U. But U is a subrepresentation, so applying the map $\rho(g)$ to the above vector produces another vector in U. Hence the image of $\rho(g) \circ \tilde{f} \circ \rho(g^{-1})$ lies in U.
Moreover, if x lies in U then $\rho(g^{-1})(x) \in U$ and then

$$\tilde{f}(\rho(g^{-1})(x)) = \rho(g^{-1})(x)$$

because $\tilde{f}$ is a projection onto U. So:

$$\rho(g) \circ \tilde{f} \circ \rho(g^{-1})(x) = \rho(g)(\rho(g^{-1})(x)) = x$$

This proves that $\rho(g) \circ \tilde{f} \circ \rho(g^{-1})$ is indeed a projection with image U, and this is true for each choice of $g \in G$. There is no reason for any of these projections to be G-linear, but we can *average* them, and define:

$$f = \frac{1}{|G|} \sum_{g \in G} \rho(g) \circ \tilde{f} \circ \rho(g^{-1})$$

By Lemma 1.5.11 this map f is another projection with image U, and we claim that this one is G-linear. We need to show that for any $h \in G$ we have:

$$f \circ \rho(h) = \rho(h) \circ f$$

The left-hand-side of this equation is the map

$$\begin{aligned} f \circ \rho(h) &= \frac{1}{|G|} \sum_{g \in G} \rho(g) \circ \tilde{f} \circ \rho(g^{-1}) \circ \rho(h) \\ &= \frac{1}{|G|} \sum_{g \in G} \rho(g) \circ \tilde{f} \circ \rho(g^{-1}h) \end{aligned}$$

and the right-hand-side is the map:

$$\rho(h) \circ f = \frac{1}{|G|} \sum_{g \in G} \rho(hg) \circ \tilde{f} \circ \rho(g^{-1})$$

But these two expressions are actually the same: we've just relabelled/permuted the group elements appearing in the sum, sending $g \mapsto hg$. So f is indeed a G-linear projection onto U, and $\mathrm{Ker}(f)$ must be a complementary subrepresentation. □

So if V contains a subrepresentation U then we can split V up as a direct sum.

Every representation V contains two obvious subrepresentations, namely V itself and the zero-dimensional subspace $\{0\} \subset V$. A subrepresentation which is not one of these two is called a *proper* subpresentation.

Definition 1.5.12. If (V, ρ) is a representation of G with no proper subrepresentations then we call it an **irreducible** representation.

The real power of Maschke's Theorem is the following corollary:

Corollary 1.5.13. *Every representation can be written as a direct sum*

$$U_1 \oplus U_2 \oplus \ldots \oplus U_r$$

of subrepresentations, where each U_i is irreducible.

Proof. Let V be a representation of G, of dimension n. If V is irreducible, we're done. If not, V contains a subrepresentation $U \subset V$, and by Maschke's Theorem,

$$V = U \oplus W$$

for some other subrepresentation W. Both W and U have dimension less than n. If they're both irreducible, we're done. If not, one of them contains a subrepresentation, so it splits as a direct sum of smaller subrepresentations. Since n is finite this process will terminate in a finite number of steps. □

So every representation is built up from irreducible representations in a straight-forward way. This makes irreducible representations very important, so we abbreviate the name and call them **irreps**. They're like the 'prime numbers' of representation theory.

Obviously any 1-dimensional representation is irreducible. Here is an example of a 2-dimensional irrep.

Example 1.5.14. Let $G = S_3$, which is the same as D_3. This group can be generated by the permutations

$$\sigma = (123) \quad \text{and} \quad \tau = (12)$$

and they have relations:

$$\sigma^3 = \tau^2 = e, \qquad \tau\sigma\tau = \sigma^{-1}$$

Let $V = \mathbb{C}^2$ and set

$$\rho(\sigma) = \begin{pmatrix} \omega & 0 \\ 0 & \omega^{-1} \end{pmatrix} \qquad \rho(\tau) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

where $\omega = e^{\frac{2\pi i}{3}}$. This defines a representation of G; it's equivalent to the representation coming from the action of D_3 on a triangle (see the Problem Sheets). Let's show that V is irreducible.

Suppose for a contradiction that $U \subset V$ is a proper subrepresentation. Then we must have $\dim U = 1$, so U must be the span of a vector $\mathbf{x} \in \mathbb{C}^2$ which is an eigenvector for both $\rho(\sigma)$ and $\rho(\tau)$. Since $\rho(\sigma)$ is diagonal the eigenvectors are obvious, they are (any multiple of) the two standard basis vectors. And neither of these is an eigenvector for $\rho(\tau)$. So V has no proper subrepresentations.

Now let's see an example of Corollary 1.5.13.

Example 1.5.15. The regular representation of $C_4 = \langle g \,|\, g^4 = 4\rangle$ is given by $V_{reg} = \mathbb{C}^4$ and

$$\rho_{reg}(g) = M = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Suppose $\mathbf{x} \in \mathbb{C}^4$ is an eigenvector of M. Then it's also an eigenvector of $\rho_{reg}(g^2) = M^2$ and $\rho_{reg}(g^3) = M^3$ so $\langle \mathbf{x} \rangle \subset \mathbb{C}^4$ is a 1-dimensional subrepresentation. The eigenvectors of M are

$$\mathbf{x}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \qquad \mathbf{x}_2 = \begin{pmatrix} 1 \\ -i \\ -1 \\ i \end{pmatrix} \qquad \mathbf{x}_3 = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} \qquad \mathbf{x}_4 = \begin{pmatrix} 1 \\ i \\ -1 \\ -i \end{pmatrix}$$

with eigenvalues $1, i, -1, -i$ respectively. So V_{reg} is the direct sum of four 1-dimensional irreps:

$$U_1 = \langle \mathbf{x}_1 \rangle \qquad U_2 = \langle \mathbf{x}_2 \rangle \qquad U_3 = \langle \mathbf{x}_3 \rangle \qquad U_4 = \langle \mathbf{x}_4 \rangle$$

In the eigenvector basis:

$$\rho_{reg}(g) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -i \end{pmatrix}$$

Look back at Examples 1.1.1, 1.1.2 and 1.5.5. In each one we took a matrix representation and found a basis in which every matrix became diagonal, *i.e.* we split each representation as a direct sum of 1-dimensional irreps.

Proposition 1.5.16. *Let (V, ρ) be a representation of G. Then V is a direct sum of 1-dimensional irreps if and only if there exists a basis of V such that every matrix $\rho(g)$ is diagonal.*

Proof. ($\Rightarrow$) Suppose

$$V = U_1 \oplus \ldots \oplus U_n$$

with each U_i a 1-dimensional subrepresentation. Pick a non-zero vector $\mathbf{x}_i$ from each U_i. Then $\{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$ is a basis for V. For any $g \in G$, the matrix $\rho(g)$ preserves $\langle \mathbf{x}_i \rangle = U_i$ for all i, so $\rho(g)$ is a diagonal matrix with respect to this basis.

($\Leftarrow$) Let $\{\mathbf{x}_1, \ldots, \mathbf{x}_n\} \subset V$ be such a basis. Then $\mathbf{x}_i$ is an eigenvector for every $\rho(g)$, so $U_i = \langle \mathbf{x}_i \rangle$ is a 1-dimensional subrepresentation, and $V = U_1 \oplus ... \oplus U_n$ because the $\mathbf{x}_i$'s form a basis. $\square$

We will see soon that if G is abelian, every representation of G splits as a direct sum of 1-dimensional irreps. When G is not abelian, this is not true.

Example 1.5.17. Let $G = S_3$ and let (V, ρ) be the permutation representation. In Example 1.5.6 we saw that V can be split up as a direct sum

$$V = U \oplus W$$

of a 2-dimensional subrepresentation U and a 1-dimensional subrepresentation W. Hence we can put our matrices in block-diagonal form with blocks of sizes 2 and 1. To go further, and make the matrices actually diagonal, we would need to split U as a direct sum of two 1-dimensional subrepresentations. But we claim that this is impossible because U is irreducible.

To verify this claim you can either:

(i) Check that the matrices

$$\rho_U(\sigma) = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} \quad \text{and} \quad \rho_U(\tau) = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}$$

have have no common eigenvector, or:

(ii) Check that these two matrices don't commute with each other. If there was a basis in which they both became diagonal matrices then they would have to commute.

A third method is to notice that U is isomorphic to the 'triangle' irrep from Example 1.5.14. Use the basis

$$\mathbf{y}_1 = \begin{pmatrix} 1 \\ -\omega \end{pmatrix} \quad \text{and} \quad \mathbf{y}_2 = \begin{pmatrix} \omega^{-1} \\ -\omega \end{pmatrix}$$

and remember the identity $1 + \omega + \omega^{-1} = 0$. The fact that this same 2-dimensional irrep appears in two different constructions is not a coincidence, as we shall learn later.

1.6. Schur's lemma and abelian groups

Theorem 1.6.1 (Schur's Lemma)**.** *Let (V, ρ_V) and (W, ρ_W) be two irreducible representations of G.*

(i) Let $f : V \to W$ be a G-linear map. Then either f is an isomorphism, or f is the zero map.
(ii) Let $f : V \to V$ be a G-linear map. Then $f = \lambda \mathbf{1}_V$ for some $\lambda \in \mathbb{C}$.

Proof. (i) Suppose f is not the zero map. $\mathrm{Ker}(f) \subset V$ is a subrepresentation of V, but V is an irrep, so either $\mathrm{Ker}(f) = 0$ or V. Since $f \neq 0$, $\mathrm{Ker}(f) = 0$, *i.e.* f is an injection. Also, $\mathrm{Im}(f) \subset W$ is a subrepresentation, and W is irreducible, so $\mathrm{Im}(f) = 0$ or W. Since $f \neq 0$, $\mathrm{Im}(f) = W$, *i.e.* f is a surjection. So f is an isomorphism.

(ii) Every linear map from V to V has at least one eigenvalue. Let λ be an eigenvalue of f and consider

$$\hat{f} = (f - \lambda \mathbf{1}_V) : V \to V$$

Then $\hat{f}$ is G-linear, because

$$\begin{aligned} \hat{f}(\rho_V(g)(x)) &= f(\rho_V(g)(x)) - \lambda \rho_V(g)(x) \\ &= \rho_V(g)(f(x)) - \rho_V(g)(\lambda x) \\ &= \rho_V(g)(\hat{f}(x)) \end{aligned}$$

for all $g \in G$ and $x \in V$. Since λ is an eigenvalue, $\mathrm{Ker}(\hat{f})$ is at least 1-dimensional. So by part 1, $\hat{f}$ is the zero map, *i.e.* $f = \lambda \mathbf{1}_V$. $\square$

Aside: (i) works over any field whereas (ii) is using the fact that $\mathbb{C}$ is algebraically closed. Schur's Lemma lets us understand the representation theory of abelian groups completely.

Proposition 1.6.2. *Suppose G is abelian. Then every irrep of G is 1-dimensional.*

Proof. Let (V, ρ) be an irrep of G. Pick any $h \in G$ and consider the linear map

$$\rho(h) : V \to V$$

In fact this is G-linear, because

$$\begin{aligned} \rho(h)(\rho(g)(x)) &= \rho(hg)(x) \\ &= \rho(gh)(x) \qquad \text{as } G \text{ is abelian} \\ &= \rho(g)(\rho(h)(x)) \end{aligned}$$

for all $g \in G, x \in V$. So by Schur's Lemma, $\rho(h) = \lambda_h \mathbf{1}_V$ for some $\lambda_h \in \mathbb{C}$. So every element of G is mapped by ρ to a multiple of $\mathbf{1}_V$. Now pick any $x \in V$. For any $h \in G$, we have

$$\rho(h)(x) = \lambda_h x \in \langle x \rangle$$

so $\langle x \rangle$ is a 1-dimensional subrepresentation of V. But V is an irrep, so $\langle x \rangle = V$, and hence V is 1-dimensional. $\square$

Corollary 1.6.3. *Let (V, ρ) be a representation of an abelian group. Then there exists a basis of V such that every $g \in G$ is represented by a diagonal matrix $\rho(g)$.*

Proof. By Maschke's Theorem, we can split ρ as a direct sum

$$V = U_1 \oplus U_2 \oplus \ldots \oplus U_n$$

of irreps. By Proposition 1.6.2, each U_i is 1-dimensional. Now apply Proposition 1.5.16. $\square$

Let's describe all the irreps of cyclic groups, which are the simplest abelian groups. Let $G = C_k = \langle g \,|\, g^k = e \rangle$. We've just proved that all irreps of G are 1-dimensional. A 1-dimensional representation of G is a homomorphism:

$$\rho : G \to GL_1(\mathbb{C})$$

This is determined by a single number

$$\rho(g) \in \mathbb{C}$$

such that $\rho(g)^k = 1$, *i.e.* $\rho(g)$ is a kth root of unity. So $\rho(g) = e^{\frac{2\pi i}{k} q}$ for some $q = [0, \ldots, k-1]$.
This gives us k possible irreps $\rho_0, \rho_1, ..., \rho_{k-1}$, defined by:

$$\rho_q : g \mapsto e^{\frac{2\pi i}{k} q}$$

Moreover all these irreps are distinct, *i.e.* ρ_q is not isomorphic to ρ_r if $r \neq q$. If they were isomorphic then they'd be equivalent (Proposition 1.4.11), but 1-dimensional matrix representations are only equivalent if they're exactly the same ($GL_1(\mathbb{C})$ is abelian so conjugating doesn't do anything).

Example 1.6.4. Let $G = C_4 = \langle g \,|\, g^4 = e \rangle$. There are four irreps of G, all 1-dimensional. They are:

$$\begin{aligned} \rho_0 &: g \mapsto 1 \quad \text{(the trivial representation)} \\ \rho_1 &: g \mapsto e^{\frac{2\pi i}{4}} = i \\ \rho_2 &: g \mapsto e^{\frac{2\pi i}{4} \times 2} = -1 \\ \rho_3 &: g \mapsto e^{\frac{2\pi i}{4} \times 3} = -i \end{aligned}$$

Look back at Example 1.1.1. We wrote down a representation

$$\rho : g \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

After diagonalising, this became the equivalent representation

$$\rho : g \mapsto \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

So ρ is the direct sum of ρ_1 and ρ_3. In Example 1.5.15 we studied the regular representation of C_4, and it found that it decomposes as the direct sum of all four irreps $\rho_0, \rho_1, \rho_2, \rho_3$.

More generally, let G be a direct product of cyclic groups

$$G = C_{k_1} \times C_{k_2} \times \ldots \times C_{k_r}$$

G is generated by elements $g_1, \ldots, g_r$ such that $g_t^{k_t} = e$ and every pair g_s, g_t commutes. An irrep of G must be a homomorphism

$$\rho : G \to GL_1(\mathbb{C})$$

and this is determined by r numbers

$$\rho(g_1), \ldots, \rho(g_r)$$

such that $\rho(g_t)^{k_t} = 1$ for all t, *i.e.* $\rho(g_t) = e^{\frac{2\pi i}{k_t} q_t}$ for some $q_t \in [0, \ldots, k_t - 1]$. This gives $k_1 \times \ldots \times k_r$ distinct 1-dimensional irreps. We label them $\rho_{q_1,\ldots,q_r}$ where:

$$\rho_{q_1,\ldots,q_r} : g_t \mapsto e^{\frac{2\pi i}{k_t} q_t}$$

Notice that the number of irreps is equal to the size of G. We'll return to this fact later.

Example 1.6.5. Let $G = C_2 \times C_2 = \langle g, h \,|\, g^2 = h^2 = e, gh = hg \rangle$. There are four irreps of G, they are:

$$\begin{aligned} \rho_{0,0} &: g \mapsto 1,\ h \mapsto 1 \quad \text{(the trivial representation)} \\ \rho_{0,1} &: g \mapsto 1,\ h \mapsto -1 \\ \rho_{1,0} &: g \mapsto -1,\ h \mapsto 1 \\ \rho_{1,1} &: g \mapsto -1,\ h \mapsto -1 \end{aligned}$$

In Example 1.1.2 we found a representation of $C_2 \times C_2$ with:

$$\rho(g) = \hat{S} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \text{and} \qquad \rho(h) = \hat{T} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

So ρ is the direct sum of $\rho_{0,1}$ and $\rho_{1,0}$.

You may have heard of the fundamental result:

Theorem (Structure theorem for finite abelian groups)**.** *Every finite abelian group is a direct product of cyclic groups.*

So now we know nearly everything about representations of finite abelian groups. Non-abelian groups are harder. But we do have one more corollary of Proposition 1.6.2 which applies to any group.

Corollary 1.6.6. *Let (V, ρ) be a representation of any group G, and let $g \in G$ be an element of order k. Then there exists a basis of V such that $\rho(g)$ is diagonal, and the diagonal entries are kth roots of unity.*

Notice the difference with the Corollary 1.6.3. With abelian groups we can diagonalize $\rho(g)$ for every $g \in G$ simultaneously. Here we are just diagonalizing one $\rho(g)$. This is not very impressive because 'almost all' matrices are diagonalizable.

Proof. Consider the subgroup $\langle g \rangle \subset G$, which is isomorphic to the cyclic group of order k, and n particular it is abelian). Restricting ρ to this subgroup gives a representation of C_k. Then Corollary 1.6.3 tells us we can find a basis of V such that $\rho(g)$ is diagonal, and since $\rho(g)^k = \mathbf{1}_V$ every diagonal entry is a kth root of unity. □

Aside: You can also prove this last result using Jordan Normal Form. If a matrix M satisifies $M^k = I_n$ then it's easy to compute that all the Jordan blocks must have size 1.

1.7. More on decomposition into irreps

In Section 1.5 we proved the basic result (Corollary 1.5.13) that every representation can be decomposed into irreps. In this section, we're going to prove that this decomposition is unique. Then we're going to look at the decomposition of the regular representation, which turns out to be very powerful.

Let V and W be vector spaces, and write

$$\mathrm{Hom}(V, W)$$

for the set of all linear maps from V to W. You should recall that $\mathrm{Hom}(V, W)$ is itself a vector space. If f_1, f_2 are two linear maps $V \to W$ then their sum is defined by

$$\begin{aligned}(f_1 + f_2) : V &\to W \\ x &\mapsto f_1(x) + f_2(x)\end{aligned}$$

and scalar multiplication is defined similarly. If we choose bases for V and W then we can write any linear map (uniquely) as a matrix, so we get a bijection:

$$\mathrm{Hom}(V, W) \cong \mathrm{Mat}_{n\times m}(\mathbb{C})$$

where $m = \dim V$ and $n = \dim W$, and this is an isomorphism of vector spaces. In particular the dimension of $\mathrm{Hom}(V, W)$ is $nm = (\dim V)(\dim W)$.

Now suppose we have a group G and two representations (V, ρ_V) and (W, ρ_W). In this situation we are usually interested in G-*linear* maps from V to W, not arbitrary linear maps. So let's write

$$\mathrm{Hom}_G(V, W) \subset \mathrm{Hom}(V, W)$$

for the set of G-linear maps. It's easy to check that this is a subspace, *i.e.* if f_1, f_2 are G-linear then so are $f_1 + f_2$, and λf_1 for any scalar $\lambda \in \mathbb{C}$. So $\mathrm{Hom}_G(V, W)$ is a vector space, and it has a dimension, which is some natural number between 0 and nm. This number is a useful thing to know.

For example, we can restate Schur's Lemma (Theorem 1.6.1) in the following way:

Proposition 1.7.1 (Schur's Lemma, v2). *Let V and W be irreps of G. Then*

$$\dim \mathrm{Hom}_G(V,W) = \begin{cases} 0 & \text{if } V \text{ and } W \text{ aren't isomorphic} \\ 1 & \text{if } V \text{ and } W \text{ are isomorphic} \end{cases}$$

Proof. Suppose V and W aren't isomorphic. Then by Schur's Lemma the only G-linear map from V to W is the zero map, so:

$$\mathrm{Hom}_G(V,W) = \{0\}$$

Alternatively, suppose that $f_0 : V \to W$ is an isomorphism. Then for any $f \in \mathrm{Hom}_G(V,W)$ consider:

$$f_0^{-1} \circ f \in \mathrm{Hom}_G(V,V)$$

By Schur's Lemma we have $f_0^{-1} \circ f = \lambda \mathbf{1}_V$, so $f = \lambda f_0$. Thus f_0 spans $\mathrm{Hom}_G(V,W)$. □

Most representations are not irreps, but (by Maschke's Theorem) they can be decomposed into direct sums of irreps. So we need to think quickly about how linear maps behave with direct sums.

Lemma 1.7.2. *Let U,V,W be three vector spaces. Then we have isomorphisms:*

(i) $\mathrm{Hom}(V, U\oplus W) \cong \mathrm{Hom}(V,U) \oplus \mathrm{Hom}(V,W)$

(ii) $\mathrm{Hom}(U\oplus W, V) \cong \mathrm{Hom}(U,V) \oplus \mathrm{Hom}(W,V)$

This says that every linear map $f : V \to U \oplus W$ is given uniquely by a pair (f_1, f_2) where $f_1 : V \to U$ and $f_2 : V \to W$ are linear. And similarly for the other direction. Also notice that all four spaces here have the same dimension, namely

$$(\dim V)(\dim W + \dim U)$$

so the statement is at least plausible.

Proof. Consider the inclusion and projection maps

$$\begin{tikzcd} U \arrow[r, shift left, "\iota_U"] & U\oplus W \arrow[l, shift left, "\pi_U"] \arrow[r, shift left, "\pi_W"] & W \arrow[l, shift left, "\iota_W"] \end{tikzcd}$$

where

$$\iota_U : x \mapsto (x,0)$$

$$\pi_U : (x,y) \mapsto x$$

and similarly for ι_W and π_W. From their definition it follows immediately that:

$$\iota_U \circ \pi_U + \iota_W \circ \pi_W = \mathbf{1}_{U\oplus W}$$

(i) Define

$$P\colon\ \mathrm{Hom}(V, U\oplus W) \to \mathrm{Hom}(V,U) \oplus \mathrm{Hom}(V,W)$$

by

$$P\colon\ f \mapsto (\pi_U \circ f, \pi_W \circ f)$$

In the other direction, define

$$Q\colon\ \mathrm{Hom}(V,U) \oplus \mathrm{Hom}(V,W) \to \mathrm{Hom}(V, U\oplus W)$$

by

$$Q\colon\ (f_U, f_W) \mapsto \iota_U \circ f_U + \iota_W \circ f_W$$

Claim 1.7.3. *P and Q are linear maps.*

Also, P and Q are inverse to each other. We check that

$$\begin{aligned} Q \circ P : f \mapsto & \iota_U \circ \pi_U \circ f + \iota_W \circ \pi_W \circ f \\ &= (\iota_U \circ \pi_U + \iota_W \circ \pi_W) \circ f \\ &= f \end{aligned}$$

and both vector spaces have the same dimension, so $P \circ Q$ must also be the identity map (or you can check this directly). So P is an isomorphism of vector spaces.
(ii) Define

$$I : \mathrm{Hom}(U \oplus W, V) \to \mathrm{Hom}(U, V) \oplus \mathrm{Hom}(W, V)$$

by

$$I : f \mapsto (f \circ \iota_U, f \circ \iota_W)$$

and

$$J : \mathrm{Hom}(U, V) \oplus \mathrm{Hom}(W, V) \to \mathrm{Hom}(U \oplus W, V)$$

by

$$J : (f_U, f_V) \mapsto f_U \circ \pi_U + f_W \circ \pi_W$$

Then use very similar arguments to those in (i). □

Corollary 1.7.4. *If U, V, W are representations of G, then we have isomorphisms*

(i) $\mathrm{Hom}_G(V, U \oplus W) \cong \mathrm{Hom}_G(V, U) \oplus \mathrm{Hom}_G(V, W)$

(ii) $\mathrm{Hom}_G(U \oplus W, V) \cong \mathrm{Hom}_G(U, V) \oplus \mathrm{Hom}_G(W, V)$

Proof. To prove (i) we just need to know that the maps P and Q from the proof of Lemma 1.7.2 take G-linear maps to G-linear maps. But this follows immediately from the fact that $\iota_U, \pi_U, \iota_W, \pi_W$ are all G-linear, and that compositions and sums of G-linear maps are all G-linear. Part (ii) is identical. □

Now that we've dealt with these technicalities we can get back to learning more about the decompostion of representations into irreps.

Proposition 1.7.5. *Let (V, ρ) be a representation, and let*

$$V = U_1 \oplus \ldots \oplus U_s$$

be a decomposition of V into irreps. Let W be any irrep of G. Then the number of irreps in the set $\{U_1, \ldots, U_s\}$ which are isomorphic to W is equal to the dimension of $\mathrm{Hom}_G(W, V)$*. It's also equal to the dimension of* $\mathrm{Hom}_G(V, W)$*.*

Proof. By Corollary 1.7.4,

$$\mathrm{Hom}_G(W, V) = \bigoplus_{i=1}^{s} \mathrm{Hom}_G(W, U_i)$$

so

$$\dim \mathrm{Hom}_G(W, V) = \sum_{i=1}^{s} \dim \mathrm{Hom}_G(W, U_i)$$

By Proposition 1.7.1, this equals the number of irreps in $\{U_1, \ldots, U_s\}$ that are isomorphic to W. An identical argument works if we consider $\mathrm{Hom}_G(V, W)$ instead. □

Now we can prove uniqueness of irrep decomposition.

Theorem 1.7.6. *Let V be a representation of G, and let*

$$V = U_1 \oplus \ldots \oplus U_s$$
$$V = \hat{U}_1 \oplus \ldots \oplus \hat{U}_r$$

be two decompositions of V into irreducible subrepresentations. Then the two sets of irreps $\{U_1, \ldots, U_s\}$ and $\{\hat{U}_1, \ldots, \hat{U}_r\}$ are the same, i.e. *$s = r$ and (possibly after reordering) U_i and $\hat{U}_i$ are isomorphic for all i.*

Proof. Let W be any irrep of G. By Proposition 1.7.5, the number of irreps in the first decomposition that are isomorphic to W is equal to $\dim \mathrm{Hom}_G(W, V)$. But the number of irreps in the second decomposition that are isomorphic to W is also equal to $\dim \mathrm{Hom}_G(W, V)$. So for any irrep W, the two decompositions contain the same number of factors isomorphic to W. □

Example 1.7.7. Let $G = S_3$. So far we've met three irreps of this group:

- (U_1, ρ_1), the 1-dimensional trivial representation.
- (U_2, ρ_2), the sign representation (see Example 1.3.6). This is also 1-dimensional.
- (U_3, ρ_3), the 2-dimensional irrep from Example 1.5.14.

For any natural numbers a, b, c we can form the representation

$$U_1^{\oplus a} \oplus U_2^{\oplus b} \oplus U_3^{\oplus c}$$

By the above theorem all of these representations are distinct, *i.e.* no two are isomorphic.

So if we could find all the irreps of a group G (up to isomorphism) then we would know all the representations of G. Each representation can be described, uniquely, as a direct sum of some number of copies of each irrep. This is similar to the relationship between integers and prime numbers: each integer can be written uniquely as a product of prime numbers, with each prime occuring with some multiplicity. However, there are infinitely many prime numbers! As we shall see shortly, the situation for representations is much simpler.

In Section 1.3 we constructed the regular representation of any group G. We take a vector space V_{reg} which has a basis

$$\{b_g \mid g \in G\}$$

(so $\dim V_{reg} = |G|$), and define

$$\rho_{reg} : G \to GL(V_{reg})$$

by

$$\rho_{reg}(h) : b_g \mapsto b_{hg}$$

and extending linearly. We claimed that this representation was very important. Here's why.

Theorem 1.7.8. *Let $V_{reg} = U_1 \oplus \ldots \oplus U_s$ be the decomposition of V_{reg} as a direct sum of irreps. Then for any irrep W of G, the number of factors in the decomposition that are isomorphic to W is equal to* $\dim W$.

Before we look at the proof let's see the most important corollaries of this result.

Corollary 1.7.9. *Any group G has only finitely many irreps (up to isomorphism), and the number of irreps is at most $|G|$.*

Proof. Any irrep W occurs in the decomposition of V_{reg} at least once, since $\dim W \geq 1$, and $\dim V_{reg} = |G|$. □

So for any group G there is a finite list $U_1, ..., U_r$ of irreps of G (up to isomorphism), and every representation of G can be written uniquely as a direct sum

$$U_1^{\oplus a_1} \oplus ... \oplus U_r^{a_r}$$

for some non-negative integers $a_1, ..., a_r$. In particular, Theorem 1.7.8 says that V_{reg} decomposes as

$$V_{reg} = U_1^{\oplus d_1} \oplus ... \oplus U_r^{\oplus d_r}$$

where:

$$d_i = \dim U_i$$

Example 1.7.10. Let $G = S_3$, and let U_1, U_2, U_3 be the three irreps of S_3 from Example 1.7.7. The regular representation V_{reg} of S_3 decomposes as

$$V_{reg} = U_1 \oplus U_2 \oplus U_3^{\oplus 2} \oplus ...$$

But $\dim V_{reg} = |S_3| = 6$, and

$$\dim(U_1 \oplus U_2 \oplus U_3^{\oplus 2}) = 1 + 1 + 2 \times 2 = 6$$

so there cannot be any other irreps of S_3.

Corollary 1.7.11. *Let $U_1, \ldots, U_r$ be all the irreps of G, and let $\dim U_i = d_i$. Then*

$$\sum_{i=1}^{r} d_i^2 = |G|$$

Proof. By Theorem 1.7.8,

$$V_{reg} = U_1^{\oplus d_1} \oplus \ldots \oplus U_r^{\oplus d_r}$$

Now take dimensions of each side. □

Notice this is consistent with out results on abelian groups. If G is abelian then $d_i = 1$ for all i (Proposition 1.6.2) so this formula says that

$$r = \sum_{i=1}^{r} d_i^2 = |G|$$

i.e. the number of irreps of G is exactly the size of G. This is what we found in Section 1.6.

The formula in Corollary 1.7.11 is remarkably powerful.

Example 1.7.12. Let $G = S_4$. Let $U_1, \ldots, U_r$ be all the irreps of G, with dimensions $d_1, \ldots, d_r$. Let U_1 be the 1-dimensional trivial representation and U_2 be the sign representation, so $d_1 = d_2 = 1$. For any symmetric group S_n these are the only possible 1-dimensional representations (see Problem Sheets), so we must have $d_i > 1$ for $i \geq 3$. We have:

$$\begin{aligned} d_1^2 + \ldots + d_r^2 &= |G| = 24 \\ \Rightarrow \quad d_3^2 + \ldots + d_r^2 &= 22 \end{aligned}$$

This has only 1 solution. Obviously $d_k \leq 4$ for all k, as $5^2 = 25$. Suppose that $d_r = 4$, then we would have

$$d_3^2 + \ldots + d_{r-1}^2 = 22 - 16 = 6$$

This is impossible, so actually $d_k \in [2,3]$ for all k. The number of k such that $d_k = 3$ must be even because 22 is even, and we can't have $d_k = 2$ for all k since $4 \nmid 22$. Therefore, the only possibility is that $d_3 = 2$ and $d_4 = 3$ and $d_5 = 3$. So G has five irreps with these dimensions.

Example 1.7.13. Let $G = D_4$. Let the irreducible representations be $U_1, \ldots, U_r$ with dimensions $d_1, \ldots, d_r$. As usual, let U_1 be the 1-dimensional trivial representation. So

$$d_2^2 + \ldots + d_r^2 = |G| - 1 = 7$$

So either

(i) $r = 8$, and $d_i = 1 \ \forall i$
(ii) $r = 5$, and $d_2 = d_3 = d_4 = 1, d_5 = 2$

In Problem Sheet 1 you show that D_4 has four 1-dimensional irreps, so in fact (ii) is true. In that question you also constructed a 2-dimensional irrep, so that must be U_5 (it's the same representation we constructed in Example 1.3.7 by thinking about the action of D_4 on a square). And now we know that there are no other irreps of D_4.

The proof of Theorem 1.7.8 follows easily from the following:

Lemma 1.7.14. *For any representation W of G, we have an isomorphism of vector spaces:*

$$\mathrm{Hom}_G(V_{reg}, W) = W$$

Proof. Recall that we have a basis vector $b_e \in V_{reg}$ corresponding to the identity element in G. Define a function

$$T : \mathrm{Hom}_G(V_{reg}, W) \to W$$

by 'evaluation at b_e', *i.e.*

$$T : f \to f(b_e)$$

Let's check that T is linear. We have

$$T(f_1 + f_2) = (f_1 + f_2)(b_e) = f_1(b_e) + f_2(b_e) = T(f_1) + T(f_2)$$

and

$$T(\lambda f) = (\lambda f)(b_e) = \lambda f(b_e) = \lambda T(f)$$

so it is indeed linear. Now let's check that T is an injection. Suppose that $f \in \mathrm{Hom}_G(V_{reg}, W)$, and that $T(f) = f(b_e) = 0$. Then for any basis vector $b_g \in V_{reg}$, we have

$$f(b_g) = f(\rho_{reg}(g)(b_e)) = \rho_W(g)(f(b_e)) = 0$$

So f sends every basis vector to zero, so it must be the zero map. Hence T is indeed an injection. Finally, we need to check that T is a surjection, so we need to show that for any $x \in W$ there is a G-linear map f from V_{reg} to W such that $f(b_e) = x$. Fix an $x \in W$, and define a linear map

$$f : V_{reg} \to W$$

by

$$f : b_g \mapsto \rho_W(g)(x)$$

Then in particular $f(b_e) = x$, so we just need to check that f is G-linear. But for any $h \in G$, we have

$$\begin{aligned} f \circ \rho_{reg}(h) :\ & b_g \mapsto \rho_W(hg)(x) \\ \rho_W(h) \circ f :\ & b_g \mapsto \rho_W(h)(\rho_W(g)(x)) = \rho_W(hg)(x) \end{aligned}$$

So $f \circ \rho_{reg}(h) = \rho_W(h) \circ f$, since both maps are linear and they agree on the basis. Thus f is indeed G-linear, and we have proved that T is a surjection. $\square$

The above lemma is surprisingly important, and we'll see it appear again later in various forms.

Proof of Theorem 1.7.8. Let $V_{reg} = U_1 \oplus \ldots \oplus U_s$ be the decomposition of the regular representation into irreps. Let W be any irrep of G. By Proposition 1.7.5, we have that $\dim \mathrm{Hom}_G(V_{reg}, W)$ equals the number of U_i that are isomorphic to W. But by Lemma 1.7.14:

$$\dim \mathrm{Hom}_G(V_{reg}, W) = \dim W$$

□

1.8. Induced representations

Suppose H is a subgroup of G. If we have a representation of G we can restrict it to H and get a representation of H (Lemma 1.3.2). Remarkably there is a way to go in the other direction, and get a representation of G from a representation of the subgroup H. These are called *induced representations*. The construction is a little complicated and we'll start with a special case.

Recall that the (left) *cosets* of the subgroup $H \subset G$ are the subsets

$$gH = \{gh \,|\, h \in H\} \subset G$$

created by left-multiplying H by different elements $g \in G$. Lagrange's Theorem tells us that:

- The cosets partition the group G, *i.e.* every element of G lies in exactly one coset.
- If $g' \in gH$ then $g'H = gH$.
- All the cosets have the same size, $|H|$, so the number of cosets is $|G|/|H|$. This number is called the *index* of H in G.

Example 1.8.1. Let $G = S_3$ and $H = \{(1), (123), (132)\}$. Since $|H| = 3$ the index is 2 so there must be two cosets. They are:

$$H = \{(1), (123), (132)\}, \qquad \text{and} \qquad \{(12), (13), (23)\}$$

If we left-multiply H by any group element we will get one of these three cosets, for example:

$$(13)H = \{(12), (13), (23)\} = (12)H = (23)H$$

A set of *coset representatives* is a list of elements $g_1, ..., g_n$, one from each coset, where n is the index of H. If we've chosen this list then we can describe the cosets as being the subsets $g_1H, ..., g_nH$. Then every group element can be written uniquely as g_jh for some $j \in [1, n]$ and some $h \in H$.

You should also recall that the group G acts on the set of cosets, via left multiplication. Let's review this quickly.

Fix a group element $g \in G$. Now take another element that lies in the ith coset, *i.e.* an element of the form g_ih. Then the product gg_ih must lie in some coset, so we can write it (uniquely) as

$$gg_ih = g_jh'$$

for some j and some $h' \in H$. Then we observe:

(i) If g_ih'' is another element of the ith coset then gg_ih'' also lies in the jth coset, since:

$$gg_ih'' = g_jh'h^{-1}h''$$

So g sends the ith coset to the jth coset.

(ii) If we take representatives of two different cosets, g_i and g_k for $i \neq k$, then gg_i and gg_k must lie in different cosets. Because if $gg_i = gg_k h$ for some h then $g_i = g_k h$.

So left-multiplying by g gives a permutation of the set of cosets. And since we've labelled our cosets we get a function

$$G \to S_n$$
$$g \mapsto \sigma_g$$

where σ_g is the permutation such that:

$$gg_i \in g_{\sigma_g(i)} H \quad \text{for each } i$$

The last step is to notice that this function from G to S_n is a homomorphism. If we take two group elements g, $\hat{g}$, which act on the cosets as permutations σ_g and $\sigma_{\hat{g}}$, then the product $g\hat{g}$ acts on the cosets as the permutation $\sigma_{g\hat{g}} = \sigma_g \circ \sigma_{\hat{g}}$.

So given a subgroup $H \subset G$ there is an associated homomorphism from G to S_n, where n is the index of H. If we compose this homomorphism with the permutation representation of S_n (as in Lemma 1.3.8) we get an n-dimensional representation of G. We'll call it:

$$\rho^H : G \to GL_n(\mathbb{C})$$

Example 1.8.2. Take $G = S_3$ and $H = C_3$ as in Example 1.8.1. We have two cosets, and we can pick representatives $g_1 = (1)$ and $g_2 = (12)$.
Now take the group element $g = (123)$. Left-multiplication by g sends:

- g_1 to g, which is in the coset $g_1 H$.
- g_2 to $(123) \circ (12) = (13)$, which is in the coset $g_2 H$.

So σ_g is the identity permutation in S_2.
If instead we take $g' = (12)$ then left-multiplication by g' sends:

- g_1 to g', which is in the coset $g_2 H$.
- g_2 to (1), which is in the coset $g_1 H$.

So σ'_g is the permutation $(12) \in S_2$.
Composing with the permutation representation of S_2 we get a representation of G:

$$\rho^H : S_3 \to GL_2(\mathbb{C})$$

with

$$\rho^H : (123) \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad \rho^H : (12) \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

This is not an irrep, it decomposes into the two 1d irreps of S_3.

In fact ρ^H is *never* an irrep (except in the trivial case $H = G$) because the permutation representation of S_n is not irreducible, as we saw in Example 1.5.7. In particular ρ^H always contains at least one copy of the trivial irrep of G.

Example 1.8.3. The dihedral group D_4 can be realised as a subgroup $H \subset S_4$ (just number the corners of your square, so every symmetry gives a permutation). Since $|S_4| = 24$ and $|D_4| = 3$ this subgroup has index 3. So it gives us a representation:

$$\rho^H : S_4 \to GL_3(\mathbb{C})$$

As we just observed, this 3-dimensional representation must split as

$$U_1 \oplus V$$

where U_1 is the trivial irrep of S_4 and V is some 2-dimensional representation. We claim that V is in fact irreducible, it is the unique 2-dimensional irrep of S_4 (see Example 1.7.12). We leave this claim as an exercise.

Example 1.8.4. Let G be any group and let H be the trivial subgroup $H = \{e\} \subset G$. This has index n where $n = |G|$, so ρ^H is an n-dimensional representation of G. But the cosets are just the elements of G, and $g \in G$ acts on them by left-multiplication, so in this case ρ^H is exactly the regular repesentation ρ_{reg} of G.

Now we move on to a more complicated version of this construction. Suppose again that $H \subset G$ is a subgroup, and suppose we have a representation of H:

$$\rho : H \to GL(V)$$

We are going to use ρ to build a representation of G called the **induced representation**. We start by taking n copies of V

$$V_1 = V_2 = ... = V_n = V$$

where n is the index of H. The vector space for the induced representation will be called $\mathrm{Ind}(V)$, and it is just the direct sum:

$$\mathrm{Ind}(V) = \bigoplus_{i=1}^{n} V_i$$

So if $\dim V = d$ then $\dim \mathrm{Ind}(V) = nd$. But we need to say how the elements of g are going to act on $\mathrm{Ind}(V)$.

Pick a set of coset representatives $g_1, ..., g_n$. For convenience we'll always assume that $g_1 = e$, so the first coset is H itself. You should think of the summands $V_i \subset \mathrm{Ind}(V)$ as being labelled by the element g_i.

Now take any element $g \in G$. We know that g acts on the cosets as some permutation σ_g, *i.e.* for each coset representative g_i we have

$$gg_i = g_{\sigma_g(i)} h_i$$

for some unique $h_i \in H$. We define a linear map

$$\mathrm{Ind}(\rho)(g) : \mathrm{Ind}(V) \to \mathrm{Ind}(V)$$

by the following rule:

- $\mathrm{Ind}(\rho)(g)$ maps any vector in the summand V_i to a vector in the summand $V_{\sigma_g(i)}$.
- The linear map $\mathrm{Ind}(\rho)(g) : V_i \to V_{\sigma_g(i)}$ is the map:

$$\rho(h_i) : V \to V$$

It is clear that $\mathrm{Ind}(\rho)(g)$ is invertible, because each $\rho(h_i)$ is invertible and σ_g is a permutation. So we have produced a function:

$$\mathrm{Ind}(\rho) : G \to GL(\mathrm{Ind}(V))$$

Lemma 1.8.5. $\mathrm{Ind}(\rho)$ *is a homomorphism,* i.e. *a representation of* G*.*

Proof. Pick two elements $g, g' \in G$. We need to verify that that two linear maps
$$\mathrm{Ind}(\rho)(g') \circ \mathrm{Ind}(\rho)(g) : \ \mathrm{Ind}(V) \to \mathrm{Ind}(V)$$
and
$$\mathrm{Ind}(\rho)(g'g) : \ \mathrm{Ind}(V) \to \mathrm{Ind}(V)$$
are equal. We can check this on each individual summand $V_i \subset V$. Let's suppose that $\sigma_g(i) = j$ and $\sigma_{g'}(j) = k$, so $\sigma_{g'g}(i) = k$. Then
$$gg_i = g_j h \quad \text{and} \quad g'g_j = g_k h'$$
for some $h, h' \in H$, and it follows that:
$$(g'g)g_i = g_k(h'h)$$
So $\mathrm{Ind}(\rho)(g'g)$ maps the summand V_i to the summand V_k, via the map $\rho(h'h)$. But $\mathrm{Ind}(\rho)(g)$ maps V_i to V_j via the map $\rho(h)$, and $\mathrm{Ind}(\rho)(g')$ maps V_j to V_k via the map $\rho(h')$. And
$$\rho(h'h) = \rho(h') \circ \rho(h)$$
since ρ is a representation. $\square$

If ρ is just the trivial 1-dimensional representation of H then $V = \mathbb{C}$, and $\mathrm{Ind}(V) = \mathbb{C}^n$, and each map $\rho(h)$ is just the identity on $\mathbb{C}$. In this case $\mathrm{Ind}(\rho)$ is exactly the representation ρ^H we studied above.

Example 1.8.6. Let $G = S_3$ and $H = C_3 \subset G$. In Example 1.8.2 we computed the representation ρ^H and found a 2-dimensional representation of S_3 which was not irreducible. This is $\mathrm{Ind}(U_1)$, where U_1 is the trival irrep of C_3.

Now let's consider the irrep U_2 of C_3, which is the 1-dimensional representation defined by
$$\rho : (123) \mapsto \omega \in \mathbb{C}^*$$
where $\omega = e^{\frac{2\pi i}{3}}$. So $\mathrm{Ind}(U_2)$ is again a 2-dimensional representation of S_3. Let's compute it.

As before fix coset representatives $g_1 = (1)$ and $g_2 = (12)$. Then for $g = (123)$ we have
$$gg_1 = (123) = g_1(123) \qquad \text{and} \qquad gg_2 = (13) = g_2(132)$$
so:
$$\mathrm{Ind}(\rho) : (123) \mapsto \begin{pmatrix} \rho(123) & 0 \\ 0 & \rho(132) \end{pmatrix} = \begin{pmatrix} \omega & 0 \\ 0 & \omega^{-1} \end{pmatrix}$$
For $g' = (12)$ we have
$$g'g_1 = (12) = g_2(1) \qquad \text{and} \qquad g'g_2 = (1) = g_1(1)$$
so:
$$\mathrm{Ind}(\rho) : (12) \mapsto \begin{pmatrix} 0 & \rho(1) \\ \rho(1) & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$
This is the 2-dimensional irrep of S_3.

When constructing $\mathrm{Ind}(V)$ we do have to make a choice of our coset representatives, and choosing different representatives will lead to different matrices. Fortunately:

Claim 1.8.7. $\mathrm{Ind}(V)$ *is independent, up to isomorphism, of the choice of coset representatives.*

Computing induced representations by hand can be tedious. But as we know, the important thing to know about a representation is not the matrices but the list of irreps in its irrep decomposition. And there is a theorem about induced representations which helps us find this information.

Theorem 1.8.8 (Frobenius reciprocity)**.** *Let $H \subset G$ be a subgroup. Let V be a representation of H, and let W be a representation of G. Then we have an isomorphism of vector spaces:*

$$\mathrm{Hom}_G(\mathrm{Ind}(V), W) \cong \mathrm{Hom}_H(V, W)$$

Remember that W also gives us a representation of H just by restricting (Lemma 1.3.2), so the expression $\mathrm{Hom}_H(V, W)$ does make sense.

Before we prove the theorem let's explain how it helps us understand $\mathrm{Ind}(V)$. Suppose W is an irrep of G. Then by Proposition 1.7.5 the number of copies of W appearing in $\mathrm{Ind}(V)$ is:

$$\dim \mathrm{Hom}_G(\mathrm{Ind}(V), W)$$

But this equals $\dim \mathrm{Hom}_H(V, W)$ by Frobenius reciprocity. In simple cases W is also an irrep of the subgroup H, and if this is true then this number also counts the number of copies of W in V.

Example 1.8.9. Let $G = S_3$ and $H = C_3 \subset S_3$. Let U_1 and U_2 be the trivial and sign irreps of S_3. If we restrict to C_3 then obviously U_1 becomes the trivial irrep of C_3, but also U_2 becomes the trivial irrep of C_3, because C_3 contains only even permutations.

Now let V be one of the two non-trivial irreps of C_3. By Frobenius reciprocity and Schur's lemma we have

$$\dim \mathrm{Hom}_G(\mathrm{Ind}(V), U_1) = \dim \mathrm{Hom}_H(V, U_1) = 0$$

and:

$$\dim \mathrm{Hom}_G(\mathrm{Ind}(V), U_2) = \dim \mathrm{Hom}_H(V, U_2) = 0$$

Thus $\mathrm{Ind}(V)$ contains no copies of U_1 or U_2. And since it is 2-dimensional it must be the 2-dimensional irrep U_3. This agrees with what we computed in Example 1.8.6.

In general W will not be an irreducible representation of H. For example, if $W = U_3$ is the 2-dimensional irrep of S_3 then W defines a 2-dimensional representation of the subgroup $H = C_3$, and since H is abelian this cannot be an irrep. Nevertheless, you can use always Frobenius reciprocity to calculate the number of copies of W in $\mathrm{Ind}(V)$ in terms of the irrep decompositions of V and W. But we'll leave this until Section 2.

We will now prove Theorem 1.8.8. First remember that if we set $H = \{e\}$ to be the trivial subgroup, and V to be the trivial 1-dimensional representation of H, then $\mathrm{Ind}(V)$ is the regular representation of H (Example 1.8.4). Moreover in this special case we have

$$\mathrm{Hom}_H(V, W) = \mathrm{Hom}(\mathbb{C}, W) = W$$

since an H-linear map is just a linear map, and a linear map from $\mathbb{C}$ to W is determined by a single vector (the image of $1 \in \mathbb{C}$). So in this case Frobenius reciprocity reduces to

$$\mathrm{Hom}_G(V_{reg}, W) \cong W$$

which was Lemma 1.7.14. So the proof of Frobenius reciprocity should be a generalization of the proof of that lemma.

Proof of Theorem 1.8.8. (Non-examinable!)
Consider the first summand $V_1 \subset \mathrm{Ind}(V)$. By our convention this summand is 'labelled' by the element $e \in G$, representing the coset $eH = H$. If we take an element $h \in H$ then it preserves this coset, and

$$\mathrm{Ind}(\rho)(h) : V_1 \to V_1$$

is exactly $\rho(h)$. So if we only consider elements of H then V_1 is identical to the representation V.
Now take a G-linear map:

$$f : \mathrm{Ind}(V) \to W$$

Restricting f to the first summand gives us a linear map:

$$f_1 : V_1 \to W$$

Since f is G-linear, it follows immediately that f_1 is H-linear (note that f_1 is not G-linear; this doesn't even make sense because most elements of G don't preserve V_1). So we have a function:

$$\begin{aligned} T :\ \mathrm{Hom}_G(\mathrm{Ind}(V), W) &\to \mathrm{Hom}_H(V, W) \\ f &\mapsto f_1 \end{aligned}$$

As before T is a linear map. Let's prove it's an injection.
Suppose $f \in \mathrm{Hom}_G(\mathrm{Ind}(V), W)$ such that $f_1 = 0$, *i.e.* $Tf = 0$. So f vanishes on the first summand $V_1 \subset \mathrm{Ind}(V)$. Consider another component V_i, labelled by the coset representative g_i. Then

$$\mathrm{Ind}(\rho)(g_i) : V_1 \to V_i$$

is the map $\rho(e)$, *i.e.* it's the identity map on V. Since f is G-linear the square

$$\begin{tikzcd} V_1 \arrow[r, "\mathrm{Ind}(\rho)(g_i)"] \arrow[d, "f_1"'] & V_i \arrow[d, "f_i"] \\ W \arrow[r, "\rho_W(g_i)"] & W \end{tikzcd}$$

commutes. But $f_1 = 0$, and $\mathrm{Ind}(\rho)(g_i)$ is the identity, so f_i must also be zero. So f vanishes on all the summands, *i.e.* f is the zero map. This proves that T is injective.
Finally we need to prove that T is a surjection. So given any $\hat{f} \in \mathrm{Hom}_H(V, W)$ we have to construct a G-linear map $f : \mathrm{Ind}(V) \to W$ such that $f_1 = \hat{f}$. Using the square above, the ith component of f must be:

$$f_i = \rho_W(g_i) \circ \hat{f}$$

Then we claim that $f = (f_1, f_2, ..., f_n)$ is indeed G linear. Given any $g \in G$, and any summand V_i, we have

$$\mathrm{Ind}(\rho)(g) = \rho(h) \colon\ V_i \to V_j$$

where j and h are such that $gg_i = g_j h$. For f to be G-linear we need the square

$$\begin{tikzcd} V_i \arrow[r, "\mathrm{Ind}(\rho)(g)"] \arrow[d, "f_i"'] & V_j \arrow[d, "f_j"] \\ W \arrow[r, "\rho_W(g)"] & W \end{tikzcd}$$

to commute. But

$$\rho_W(g) \circ f_i = \rho_W(g) \circ \rho_W(g_i) \circ \hat{f} = \rho_W(g_j) \circ \rho_W(h) \circ \hat{f} = \rho_W(g_j) \circ \hat{f} \circ \rho(h) = f_j \circ \rho(h)$$

because ρ_W is a homomorphism and $\hat{f}$ is H-linear. $\square$

1.9. Duals and Homs

Let V be a vector space. Recall the definition of the **dual vector space**:

$$V^* = \mathrm{Hom}(V, \mathbb{C})$$

This is a special case of $\mathrm{Hom}(V, W)$ where $W = \mathbb{C}$. So $\dim V^* = \dim V$, and if $\{b_1, \ldots, b_n\}$ is a basis for V, then there is a **dual basis** $\{\beta_1, \ldots, \beta_n\}$ for V defined by

$$\beta_i(b_j) = \begin{cases} 1 & \text{if} \quad i = j \\ 0 & \text{if} \quad i \neq j \end{cases}$$

Using these bases we can identify both V and V^* with $\mathbb{C}^n$. But if you think of V as the space of column vectors then it's sometimes helpful to think of V^* as the space of *row* vectors, because each row vector defines a linear map from column vectors to $\mathbb{C}$. The formula above matches how the standard basis for row vectors acts on the standard basis for column vectors.

Now suppose we have a second vector space W and a linear map $f : V \to W$. Recall that there is a **dual linear map**:

$$f^* : W^* \to V^*$$
$$\phi \mapsto \phi \circ f$$

If we choose bases for V and W then we can write f as a matrix M. Using the dual bases for V^* and W^* we can also write f^* as a matrix; this gives us the *transpose* matrix $M^\top$.

Notice that f^* goes 'in the opposite direction' to f. Also if we have a second linear map $g : W \to U$ then it's easy to check that:

$$(g \circ f)^* = f^* \circ g^*$$

If we write everything as matrices this is the simple fact that $(NM)^\top = M^\top N^\top$.

Now let (V, ρ) be a representation. For every group element g we have a linear map

$$\rho(g) : V \to V$$

so we have a dual linear map:

$$\rho(g)^* : V^* \to V^*$$

You might hope that this defines a representation of G on V^*. But there is a problem: if we take $g, h \in G$ we have

$$\rho(gh)^* = (\rho(g) \circ \rho(h))^* = \rho(h)^* \circ \rho(h)^*$$

so the map $g \mapsto \rho(g)^*$ is not a homomorphism, it reverses multiplication instead of preserving it. Fortunately there is a way to repair this.

Definition 1.9.1. Given (V, ρ), the **dual represention** is defined by the vector space V^* and the homomorphism:

$$\rho^* : G \to GL(V^*)$$
$$g \mapsto \rho(g^{-1})^*$$

If we pick a basis and write everything as matrices, so $\rho : G \to GL_n(\mathbb{C})$, then it's easy to say what ρ^* is. It's the representation we get by composing ρ with the homomorphism:

$$GL_n(\mathbb{C}) \to GL_n(\mathbb{C})$$
$$M \mapsto M^{-\top}$$

If we just transpose the matrices it won't work, or if we just invert them all it won't work, but when we do both at the same time then it works.

Example 1.9.2. Let $G = S_3 = \langle g, h \mid g^3 = h^2 = e,\ hgh = g^{-1}\rangle$ and let (V, ρ) be the 2-dimensional irrep of G. In the appropriate basis (see Problem Sheets):

$$\rho(g) = \begin{pmatrix} \omega & 0 \\ 0 & \omega^{-1} \end{pmatrix} \quad \text{where } \omega = e^{\frac{2\pi i}{3}}$$

$$\rho(h) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

The dual representation (in the dual basis) is:

$$\rho^*(g) = \begin{pmatrix} \omega^{-1} & 0 \\ 0 & \omega \end{pmatrix}$$

$$\rho(h) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

This is equivalent to ρ under the change of basis:

$$P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

So in this case, V^* and V are isomorphic representations.

Example 1.9.3. Let $G = C_3 = \langle g \mid g^3 = e\rangle$ and consider the 1-dimensional representation:

$$\rho_1 : g \mapsto \omega = e^{\frac{2\pi i}{3}}$$

The dual representation is:

$$\rho_1^* : g \mapsto \omega^{-1} = e^{\frac{4\pi i}{3}}$$

So in this case

$$\rho_1^* = \rho_2$$

and in particular ρ_1 and ρ_1^* are not isomorphic.

So sometimes (V^*, ρ^*) is isomorphic to (V, ρ), and sometimes it is not. However:

Lemma 1.9.4. *Let $\rho : G \to GL(V)$ be a representation. Then $(V^*)^*$ is isomorphic to V as a representation.*

Proof. Pick a basis for V, and pick any $g \in G$. Then using our basis we can represent $\rho(g)$ as some matrix M. We have a dual basis for V^* and using that basis $\rho^*(g)$ is the matrix M^{-T}. Then $(V^*)^*$ also has a basis (the dual basis to the basis we have for V^*) and using it we get the matrix:

$$(\rho^*)^*(g) = \left(M^{-T}\right)^{-T} = M$$

So we have bases such that ρ and $(\rho^*)^*$ give identical matrix representations. ☐

Proposition 1.9.5. *Let V be a representation of G. Then V is irreducible if and only if V^* is irreducible.*

Proof. Suppose V is not irreducible. Then (by Maschke's Theorem) we can pick a basis for V such that every matrix $\rho(g)$ is block-diagonal. But then every matrix $\rho(g)^{-\top}$ is also block-diagonal, so V^* is not irreducible. By the same argument, if V^* is not irreducible then neither is $(V^*)^* = V$. ☐

So 'taking duals' gives an order-2 permutation of the set of irreps of G.

Now we're going to generalize this construction from the dual space to other spaces of linear maps. Take any two vector spaces V, W and consider the vector space $\mathrm{Hom}(V, W)$ of all linear maps between them. If $\{a_1, \ldots, a_n\}$ is a basis for V, and $\{b_1, \ldots, b_m\}$ is a basis for W, then we can get a basis for $\mathrm{Hom}(V, W)$ by setting:

$$\begin{aligned} f_{ji} : V &\to W \\ a_k &\mapsto \begin{cases} b_j & \text{if } k = i \\ 0 & \text{if } k \neq i \end{cases} \end{aligned}$$

The set $\{f_{ji} \,|\, i \in [1, n],\ j \in [1, m]\}$ is a basis for $\mathrm{Hom}(V, W)$. If we use our chosen bases to identify $\mathrm{Hom}(V, W)$ with $\mathrm{Mat}_{m\times n}(\mathbb{C})$ then the maps f_{ji} become the matrices which have one of their entries equal to 1 and all other entries equal to zero. This is obviously a basis.

Example 1.9.6. Let $V = W = \mathbb{C}^2$, equipped with the standard basis. Then:

$$\mathrm{Hom}(V, W) = \mathrm{Mat}_{2\times 2}(\mathbb{C})$$

This is a 4-dimensional vector space. The obvious basis is:

$$f_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad f_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad f_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

Now suppose that we have representations (V, ρ_V) and (W, ρ_W) of G. We claim that there is a natural representation of G on the vector space $\mathrm{Hom}(V, W)$. For $g \in G$, we define:

$$\begin{aligned} \rho_{\mathrm{Hom}}(g)\colon \mathrm{Hom}(V, W) &\to \mathrm{Hom}(V, W) \\ f &\mapsto \rho_W(g) \circ f \circ \rho_V(g^{-1}) \end{aligned}$$

Notice that if we set $W = \mathbb{C}$, and ρ_W to be the trivial representation then this formula reduces to the dual representation V^*. In particular we shouldn't be surprised by the appearance of g^{-1}.

We need to check the formula above does indeed define a representation.

Claim 1.9.7. *For each $g \in G$, the function $\rho_{Hom}(g)$ is a linear map from* $\mathrm{Hom}(V, W)$ *to* $\mathrm{Hom}(V, W)$.

Then we still need to check that:

(i) For all g, the linear map $\rho_{Hom}(g)$ is invertible.
(ii) The function $g \mapsto \rho_{Hom}(g)$ is a homomorphism.

Observe that

$$\begin{aligned} \rho_{Hom}(h) \circ \rho_{Hom}(g) : f \mapsto{} & \rho_W(h) \circ (\rho_W(g) \circ f \circ \rho_V(g^{-1})) \circ \rho_V(h^{-1}) \\ &= \rho_W(hg) \circ f \circ \rho_V(g^{-1}h^{-1}) \\ &= \rho_{Hom}(hg)(f) \end{aligned}$$

In particular,

$$\begin{aligned} \rho_{Hom}(g) \circ \rho_{Hom}(g^{-1}) &= \rho_{Hom}(e) \\ &= \mathbf{1}_{\mathrm{Hom}(V,W)} \\ &= \rho_{Hom}(g^{-1}) \circ \rho_{Hom}(g) \end{aligned}$$

So $\rho_{Hom}(g^{-1})$ is inverse to $\rho_{Hom}(g)$. So our formula does define a function

$$\rho_{Hom} : G \to GL(\mathrm{Hom}(V, W))$$

and it's a homomorphism, so we indeed have a representation.

Suppose we pick bases for V and W, so ρ_V and ρ_W become matrix representations:

$$\rho_V : G \to GL_n(\mathbb{C}) \qquad \text{and} \qquad \rho_W : G \to GL_m(\mathbb{C})$$

Then we can identify $\mathrm{Hom}(V, W)$ with $\mathrm{Mat}_{n\times m}(\mathbb{C})$. If $\rho_V(g) = N$, and $\rho_W(g) = M$, then

$$\rho_{Hom}(g)\colon\ \mathrm{Mat}_{n\times m}(\mathbb{C}) \to \mathrm{Mat}_{n\times m}(\mathbb{C})$$

is the linear map:

$$F \mapsto MFN^{-1}$$

Notice that this is a linear map from a vector space of dimension mn to itself. So if we want to write down a matrix for $\rho_{Hom}(g)$ then the matrix will have size $nm \times nm$.

Example 1.9.8. Let $G = C_2 = \langle g \,|\, g^2 = e\rangle$. Let $V = \mathbb{C}^2$ be the regular representation and W be the 2-dimensional trivial representation. So:

$$\rho_V(g) = N = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \text{and} \qquad \rho_W(g) = M = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Then $\mathrm{Hom}(V, W) = \mathrm{Mat}_{2\times 2}(\mathbb{C})$, and $\rho_{Hom}(g)$ is the linear map:

$$\begin{aligned} \rho_{Hom}(g)\colon\ \mathrm{Mat}_{2\times 2}(\mathbb{C}) &\to \mathrm{Mat}_{2\times 2}(\mathbb{C}) \\ F &\mapsto MFN^{-1} = F\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \end{aligned}$$

$(\mathrm{Hom}(V, W), \rho_{Hom})$ is a 4-dimensional representation of C_2. So if we choose a basis for $\mathrm{Hom}(V, W)$, we will get a 4-dimensional matrix representation:

$$\rho_{Hom} : C_2 \to GL_4(\mathbb{C})$$

Let's use our standard basis for $\mathrm{Hom}(V, W)$. It's easy to compute that:

$$\begin{aligned} \rho_{Hom}(g) : \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} &\mapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} &\mapsto \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} &\mapsto \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} &\mapsto \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \end{aligned}$$

So in this basis, $\rho_{Hom}(g)$ is given by the matrix:

$$\mathfrak{M} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Now let's consider the set of G-linear maps from V to W, which forms a subspace $\mathrm{Hom}_G(V,W) \subset \mathrm{Hom}(V,W)$. How is this subspace related to the representation ρ_{Hom}? A map f is G-linear, by definition, if it satisfes

$$\rho_W(g) \circ f = f \circ \rho_V(g)$$

for all $g \in G$. This says precisely:

$$\rho_{\mathrm{Hom}}(g)(f) = \rho_V(g) \circ g \circ \rho_W(g^{-1}) = f$$

So f is G-linear iff it is fixed by $\rho_{\mathrm{Hom}}(g)$ for every $g \in G$.

Definition 1.9.9. Let (V, ρ) be any representation. We define the **invariant subrepresentation**

$$V^G \subset V$$

to be the set

$$\{x \in V \mid \ \rho(g)(x) = x, \quad \forall g \in G\}$$

It's easy to check that V^G is actually a subspace of V, and it's obvious that it's also a subrepresentation (which justifies the name). It's isomorphic to a trivial representation.

The observation we made above is:

Lemma 1.9.10. *Let (V, ρ_V) and (W, ρ_W) be representations of G. Then*

$$\mathrm{Hom}_G(V,W) \ \subset \ \mathrm{Hom}(V,W)$$

is exactly the invariant subrepresentation $\mathrm{Hom}(V,W)^G$ *of* $\mathrm{Hom}(V,W)$

Example 1.9.11. As in Example 1.9.8 let $G = C_2$, $V = \mathbb{C}^2$ be the regular representation and $W = \mathbb{C}^2$ be the 2-dimensional trivial representation. Then

$$F \ \in \mathrm{Hom}(V,W) = \mathrm{Mat}_{2\times 2}(\mathbb{C})$$

is G-linear iff it satisfies the equation

$$\rho_W(\tau) \circ F = F \circ \rho_V(g)$$

i.e:

$$F \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = F$$

The space of matrices satisfying this equation is 2-dimensional, it's given by:

$$\mathrm{Hom}_G(V,W) = \left\{ \begin{pmatrix} a & a \\ b & b \end{pmatrix}, \ a,b \in \mathbb{C} \right\} = \left\langle \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \right\rangle$$

Alternatively we can use the standard basis to identify:

$$\mathrm{Mat}_{2\times 2}(\mathbb{C}) \cong \mathbb{C}^4$$

Then $\rho_{Hom}(g)$ is the 4×4 matrix $\mathfrak{M}$ we computed in Example 1.9.8. And $\mathbf{x} \in \mathbb{C}^4$ is in the invariant subrepresentation iff

$$\rho_{Hom}(g)(\mathbf{x}) = \mathfrak{M}\mathbf{x} = \mathbf{x}$$

which says that $\mathbf{x}$ is an eigenvector of $\mathfrak{M}$ with eigenvalue 1. It's easy to compute that this eigenspace is spanned by

$$\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} \quad \in \mathbb{C}^4$$

which matches our description of $\mathrm{Hom}_G(V,W)$ above.

In the proof of Maschke's Theorem, the key step was a clever 'averaging' trick to turn a randomly-chosen linear map into a G-linear map. This trick works more generally.

Proposition 1.9.12. *Let (V, ρ) be a representation of G. Consider the linear map*

$$\begin{aligned}\Psi\colon V &\to V\\ x &\mapsto \frac{1}{|G|}\sum_{g\in G}\rho(g)(x)\end{aligned}$$

Then Ψ is a G-linear projection from V onto V^G.

Proof. First we need to check that $\Psi(x) \in V^G$ for all x. For any $h \in G$ we have:

$$\begin{aligned}\rho(h)(\Psi(x)) &= \frac{1}{|G|}\sum_{g\in G}\rho(h)\rho(g)(x)\\ &= \frac{1}{|G|}\sum_{g}\rho(hg)(x)\\ &= \frac{1}{|G|}\sum_{g}\rho(g)(x) \qquad \text{(relabelling } g \mapsto h^{-1}g)\\ &= \Psi(x)\end{aligned}$$

Next we check that Ψ is the identity function on V^G. Let $x \in V^G$. Then:

$$\begin{aligned}\Psi(x) &= \frac{1}{|G|}\sum_{g}\rho(g)(x)\\ &= \frac{1}{|G|}\sum_{g}x = x\end{aligned}$$

Finally we check that Ψ is G-linear. For $h \in G$ we have:

$$\begin{aligned}\Psi(\rho(h)(x)) &= \frac{1}{|G|}\sum_{g\in G}\rho(g)(h)(x)\\ &= \frac{1}{|G|}\sum_{g\in G}\rho(gh)(x)\\ &= \frac{1}{|G|}\sum_{g\in G}\rho(hg)(x) \qquad \text{(relabelling } g \mapsto hgh^{-1})\\ &= \rho(h)\Psi(x)\end{aligned}$$

$\square$

As a special case, let V be a representations of G and consider the representation $\mathrm{Hom}(V, V)$. The above proposition gives us a projection

$$\Psi : \mathrm{Hom}(V, V) \to \mathrm{Hom}(V, V)$$

onto the invariant subrepresentation $\mathrm{Hom}(V, V)^G = \mathrm{Hom}_G(V, V)$. If you look back at the proof of Maschke's Theorem you'll see that we used exactly this map.

1.10. Tensor products

In this section we're going to define tensor products of representations. There are several ways to define these, of varying degrees of sophistication. We'll start with a special case which quite a lot easier.

Let (W, ρ_W) be any representation of G, and pick a basis for V so we can write it as a matrix representation:

$$\rho_W : G \to GL_n(\mathbb{C})$$

Now let (V, ρ_V) be a 1-dimensional representation. So for each $g \in G$, $\rho_V(g) \in GL_1(\mathbb{C})$ is just a complex number.

We define a new matrix representation, which we'll denote by $\rho_{V\otimes W}$, by setting:

$$\rho_{V\otimes W}(g) = \rho_V(g)\rho_W(g) \ \in GL_n(\mathbb{C})$$

This is called the *tensor product* of ρ_V and ρ_W.

Notice that all we're doing here is multiplying a matrix $\rho_W(g)$ by a scalar $\rho_V(g)$, so it's easy to see that $\rho_{V\otimes W}$ really is another representation.

If ρ_V was another n-dimensional matrix then we could multiply the $n \times n$-matrices $\rho_V(g)$ and $\rho_W(g)$ together, but this would *not* define a representation (check it!).

Example 1.10.1. Let $G = S_3$ and W be the 2-dimensional irrep with our usual choice of basis, so:

$$\rho_W(g) = \begin{pmatrix} \omega & 0 \\ 0 & \omega^{-1} \end{pmatrix}, \quad \rho_W(h) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Let V be the sign representation, so:

$$\rho_V(g) = 1, \quad \rho_V(h) = -1$$

Then $\rho_{V\otimes W}$ is given by

$$\rho_{V\otimes W}(g) = \begin{pmatrix} \omega & 0 \\ 0 & \omega^{-1} \end{pmatrix}, \quad \rho_{V\otimes W}(h) = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

Claim 1.10.2. *For any representation (W, ρ_W) and any 1-dimensional representation (V, ρ_V) we have that $\rho_{V\otimes W}$ is irreducible iff ρ_W is irreducible.*

This can be a useful way to produce new irreps out of ones we already know.

In particular, in the above example the 2-dimensional representation $\rho_{V\otimes W}$ is irreducible. But we know that there's only one 2-dimensional irrep of S_3, so $\rho_{V\otimes W}$ must be isomorphic to ρ_W. Can you find the change-of-basis matrix?

Now we'll move on to the case when $\dim V > 1$. So, let V and W be two vector spaces (of any dimension) and choose bases $\{a_1, \ldots, a_n\}$ for V and $\{b_1, \ldots, b_m\}$ for W.

Definition 1.10.3. The **tensor product** of V and W is the vector space which has a basis given by the set of symbols:

$$\big\{a_i \otimes b_t \ \mid \ i \in [1, n], \ t \in [1, m]\big\}$$

We write the tensor product of V and W as:

$$V \otimes W$$

By definition:

$$\dim(V \otimes W) = (\dim V)(\dim W)$$

Compare this to the direct sum, which has $\dim V \oplus W = \dim V + \dim W$.

If we have vectors $x \in V$ and $y \in W$, we can define a vector

$$x \otimes y \in V \otimes W$$

as follows. Write x and y in the given bases, so

$$x = \lambda_1 a_1 + \ldots + \lambda_n a_n$$
$$y = \mu_1 b_1 + \ldots + \mu_m b_m$$

for some coefficients $\lambda_i, \mu_t \in \mathbb{C}$. Then we define

$$x \otimes y = \sum_{\substack{i\in[1,n] \\ t\in[1,m]}} \lambda_i \mu_t \, a_i \otimes b_t$$

(think of expanding out the brackets). However, not all vectors in $V \otimes W$ can be written in this form.

Now let V and W carry representations of G. We can define a representation of G on $V \otimes W$, called the **tensor product representation**. We let

$$\rho_{V\otimes W}(g) : V \otimes W \to V \otimes W$$

be the linear map defined by:

$$\rho_{V\otimes W}(g) : a_i \otimes b_t \mapsto \rho_V(g)(a_i) \otimes \rho_W(g)(b_t)$$

Suppose $\rho_V(g)$ is described by the matrix M (in this given basis), and $\rho_W(g)$ is described by the matrix N. Then

$$\begin{aligned} \rho_{V\otimes W}(g) : a_i \otimes b_t \mapsto & \left(\sum_{j=1}^{n} M_{ji} a_j\right) \otimes \left(\sum_{s=1}^{m} N_{st} b_s\right) \\ &= \sum_{\substack{j\in[1,n] \\ s\in[1,m]}} M_{ji} N_{st} \, a_j \otimes b_s \end{aligned}$$

So $\rho_{V\otimes W}(g)$ is described by the $nm \times nm$ matrix $M \otimes N$, whose entries are

$$[M \otimes N]_{js,it} = M_{ji} N_{st}$$

This notation can be quite confusing! We've constructed a new matrix out of M and N by multiplying every element of M by every element of N. The new matrix has $n \times m$ rows and $n \times m$ columns. To specify a row we have to give a pair of numbers (j, s), where $j \in [1, n]$ and $s \in [1, m]$, and when we write js above we mean this pair of numbers, we don't mean their product. Similiarly to specify a column we have to give another pair of numbers (i, t). Fortunately we won't have to use this notation much.

Notice that if V is 1-dimensional then M is just a number and $M \otimes N$ is just the matrix N multiplied by the scalar M. This was the special case that we started with.

Example 1.10.4. Let $G = S_3$. Let V be the 2-dimensional irrep (with our usual basis) and let W be the direct sum of the trivial and sign irreps. So

$$\rho_V(g) = \begin{pmatrix} \omega & 0 \\ 0 & \omega^{-1} \end{pmatrix} \qquad \rho_V(h) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$
$$\rho_W(g) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \rho_W(h) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Then:

$$\rho_{V\otimes W}(g) = \begin{pmatrix} \omega\begin{pmatrix}1&0\\0&1\end{pmatrix} & 0\begin{pmatrix}1&0\\0&1\end{pmatrix} \\ 0\begin{pmatrix}1&0\\0&1\end{pmatrix} & \omega^{-1}\begin{pmatrix}1&0\\0&1\end{pmatrix}\end{pmatrix} \qquad \rho_{V\otimes W}(h) = \begin{pmatrix} 0\begin{pmatrix}1&0\\0&-1\end{pmatrix} & 1\begin{pmatrix}1&0\\0&-1\end{pmatrix} \\ 1\begin{pmatrix}1&0\\0&-1\end{pmatrix} & 0\begin{pmatrix}1&0\\0&-1\end{pmatrix}\end{pmatrix}$$

$$= \begin{pmatrix}\omega&0&0&0\\0&\omega&0&0\\0&0&\omega^{-1}&0\\0&0&0&\omega^{-1}\end{pmatrix} \qquad = \begin{pmatrix}0&0&1&0\\0&0&0&-1\\1&0&0&0\\0&-1&0&0\end{pmatrix}$$

Notice that this representation cannot possibly be irreducible because S_3 has no 4-dimensional irreps (compare this with Claim 1.10.2).

We have not checked yet that $\rho_{V\otimes W}$ is a homomorphism. However, there is a more fundamental question: how do we know that this construction is independent of our choice of bases? Both questions are answered by the following.

Proposition 1.10.5. $\mathrm{Hom}(V, W)$ *is isomorphic to* $V^* \otimes W$ *as a representation.*

And hence:

Corollary 1.10.6. $V \otimes W$ *is isomorphic to* $\mathrm{Hom}(V^*, W)$*.*

Proof.

$$\mathrm{Hom}(V^*, W) \cong (V^*)^* \otimes W \cong V \otimes W$$

by Lemma 1.9.4. □

We can view this corollary as an alternative definition for $V \otimes W$. It's better because it doesn't require us to choose bases for our vector spaces, but it's less explicit.

Aside: this definition only works for finite-dimensional vector spaces. There are other basis-independent definitions that work in general, but they're even more abstract.

Proof of Proposition 1.10.5. Let $\{\alpha_1, \ldots, \alpha_n\}$ be the basis for V^* dual to $\{a_1, \ldots, a_n\}$. Then $V^* \otimes W$ has a basis:

$$\big\{\alpha_i \otimes b_t \;\mid\; i \in [1, n],\ t \in [1, m]\big\}$$

Now recall that the space $\mathrm{Hom}(V, W)$ has a basis

$$\big\{f_{ti} \;\mid\; i \in [1, n],\ t \in [1, m]\big\}$$

where f_{it} is the linear map:

$$\begin{aligned} f_{ti} : a_i &\mapsto b_t \\ a_{\neq i} &\mapsto 0 \end{aligned}$$

This corresponds to the obvious basis for the space of matrices $\mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^m)$. So we can define an isomorphism of vector spaces between $V^* \otimes W$ and $\mathrm{Hom}(V, W)$ by mapping:

$$\alpha_i \otimes b_t \mapsto f_{ti}$$

for each i, t (if you think of the b_t's as the standard basis for column vectors $\mathbb{C}^m$, and the α_i's as the standard basis for row vectors $\mathbb{C}^n$, what we're doing here is matrix multiplication). To prove the proposition it's sufficient to check that the representation ρ_{Hom} agrees with the definition of $\rho_{V^*\otimes W}$ when we write it in the basis $\{f_{ti}\}$.

Pick $g \in G$ and let $\rho_V(g)$ and $\rho_W(g)$ be described by matrices M and N in the given bases. Then in the dual basis $\rho_V^*(g)$ is represented by the matrix M^{-T}. By definition we have

$$\rho_{\mathrm{Hom}}(g)\colon\ f_{ti} \ \mapsto\ \rho_W(g) \circ f_{ti} \circ \rho_V(g^{-1})$$

and

$$\rho_V(g^{-1})\colon\ a_k \ \mapsto\ \sum_{j=1}^{n} (M^{-1})_{jk} a_j$$

so

$$f_{ti} \circ \rho_V(g^{-1})\colon\ a_k \ \mapsto\ (M^{-1})_{ik} b_t$$

and:

$$\rho_W(g) \circ f_{ti} \circ \rho_V(g^{-1})\colon\ a_k \ \mapsto\ (M^{-1})_{ik} \left(\sum_{j=1}^{m} N_{st} b_s \right)$$

Therefore, if we write $\rho_W(g) \circ f_{ti} \circ \rho_V(g^{-1})$ in terms of the basis $\{f_{sj}\}$, we have

$$\rho_W(g) \circ f_{ti} \circ \rho_V(g^{-1}) = \sum_{\substack{j \in [1,n] \\ s \in [1,m]}} (M^{-1})_{ij} N_{st} f_{sj} = \sum_{\substack{j \in [1,n] \\ s \in [1,m]}} (M^{-T})_{ji} N_{st} f_{sj}$$

since both sides agree on each basis vector a_k. This is exactly the formula for the tensor product representation $\rho_{V^* \otimes W}$. □

In general tensor products are annoying to calculate explicitly. One of the many uses of the theory of *characters* is that they mean we don't have to do it.

2. Characters

2.1. Basic properties

Let M be an $n \times n$ matrix. Recall that the **trace** of M is

$$\mathrm{Tr}(M) = \sum_{i=1}^{n} M_{ii}$$

If N is another $n \times n$ matrix, then

$$\mathrm{Tr}(NM) = \sum_{i,j=1}^{n} N_{ij} M_{ji} = \mathrm{Tr}(MN)$$

which implies that

$$\mathrm{Tr}(P^{-1}MP) = \mathrm{Tr}(PP^{-1}M) = \mathrm{Tr}(M)$$

Definition 2.1.1. Let V be a vector space, and

$$f : V \to V$$

a linear map. Pick a basis for V and let M be the matrix describing f in this basis. We define

$$\mathrm{Tr}(f) = \mathrm{Tr}(M)$$

This definition does not depend on the choice of basis, because choosing a different basis will produce a matrix which is conjugate to M, and hence has the same trace.

Now let G be a group, and let ρ be a representation

$$\rho : V \to GL(V)$$

on a vector space V.

Definition 2.1.2. The **character** of the representation ρ is the function

$$\begin{aligned} \chi_\rho : & G \to \mathbb{C} \\ & g \mapsto \mathrm{Tr}\,(\rho(g)) \end{aligned}$$

Notice that χ_ρ is not a homomorphism in general, since generally:

$$\mathrm{Tr}(MN) \neq \mathrm{Tr}(M)\,\mathrm{Tr}(N)$$

In fact the only case when χ_ρ is a homomorphism is when $\dim V = 1$. The character of a 1-dimensional representation is the same thing as the representation itself.

Note on terminology: In some contexts the word 'character' is used to mean '1-dimensional representation'. This is a special case of our definition.

Example 2.1.3. Let $G = C_2 \times C_2 = \langle g, h \mid g^2 = h^2 = e, gh = hg \rangle$. Let ρ be the direct sum of $\rho_{1,0}$ and $\rho_{1,1}$, so

$$\rho(e) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \rho(g) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$
$$\rho(h) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \rho(gh) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

Then

$$\begin{aligned} \chi_\rho : e &\mapsto 2 \\ g &\mapsto -2 \\ h &\mapsto 0 \\ gh &\mapsto 0 \end{aligned}$$

Proposition 2.1.4. *Isomorphic representations have the same character.*

Proof. In Proposition 1.4.11 we showed that if two representations are isomorphic, then there exist bases in which they are described by the same matrix representation. □

Later on we'll prove the converse to this statement, which is quite striking: if two representations have the same character then they must be isomorphic.

Proposition 2.1.5. *Let $\rho : G \to GL(V)$ be a representation of dimension d, and let χ_ρ be its character. Then:*

(i) $\chi_\rho(e) = d$

(ii) If g and h are conjugate in G then:

$$\chi_\rho(g) = \chi_\rho(h)$$

(iii) If $g \in G$ has order k then

$$\chi_\rho(g) = \lambda_1 + ... + \lambda_d$$

where each λ_i is a kth root of unity.

(iv) For any $g \in G$:

$$\chi_\rho\left(g^{-1}\right) = \overline{\chi_\rho(g)}$$

Proof. (i) In any basis, $\rho(e)$ is the $d \times d$ identity matrix.
(ii) Suppose $g = x^{-1}hx$ for some $x \in G$. Then

$$\rho(g) = \rho(x^{-1})\rho(h)\rho(x)$$

So in any basis, the matrices for $\rho(g)$ and $\rho(h)$ are conjugate, so

$$\mathrm{Tr}\left(\rho(g)\right) = \mathrm{Tr}\left(\rho(h)\right)$$

This says that χ_ρ is a **class function**, more on these later.
(iii) Corollary 1.6.6 tells us that we can pick a basis for V such that $\rho(g)$ becomes a diagonal matrix D, which each diagonal value a kth root of unity.
(iv) In the same basis, $\rho(g^{-1})$ is represented by D^{-1}, which is diagonal with entries λ_i^{-1}. But

$$\lambda_i^{-1} = \overline{\lambda_i}$$

because a kth root of unity must have $|\lambda_i| = 1$. So:

$$\chi_\rho(g^{-1}) = \sum_{i=1}^{d} \overline{\lambda_i} = \overline{\chi_\rho(g)}$$

□

Property (iii) is already enough to show:

Corollary 2.1.6. *Let (V, ρ) be a representation of G of dimension d, and let χ_ρ be its character. Then for any $g \in G$, if $\chi_\rho(g) = d$ then we must have $\rho(g) = \mathbf{1}_V$.*

So if you know χ_ρ then you know the kernel of ρ. In particular you know whether or not ρ is faithful.

Proof. By Proposition 2.1.5(iii) we have

$$\chi_\rho(g) = \lambda_1 + ... + \lambda_d$$

with each λ_i having $|\lambda_i| = 1$. By the triangle inequality we have

$$|\chi_\rho(g)| \leq |\lambda_1| + ... + |\lambda_d| = d$$

with equality iff all the λ_i's are equal. If $\chi_\rho(g) = d$ then we must have $\lambda_i = 1$ for all i, which says that $\rho(g)$ is the identity matrix. □

The character of the regular representation ρ_{reg} is called the **regular character**, we write it as χ_{reg}.

Lemma 2.1.7.

$$\chi_{reg}(g) = \begin{cases} |G| & \text{if} \quad g = e \\ 0 & \text{if} \quad g \neq e \end{cases}$$

Proof. $\chi_{reg}(e) = \dim V_{reg} = |G|$ by Proposition 2.1.5(i). Suppose $g \neq e$. Then for all $h \in G$ we have:

$$gh \neq h$$

The regular representation has a basis $\{b_h \mid h \in G\}$, and $\rho_{reg}(g)$ is the linear map:

$$b_h \mapsto b_{gh}$$

Since $b_{gh} \neq b_h \, \forall \, h$, we have $\mathrm{Tr}\left(\rho_{reg}(g)\right) = 0$. □

This lemma can be generalized to arbitrary permutation representations (see Problem Sheets).

Now let's think about characters behave when we combine representations in various ways. Given any two functions ξ, ζ from G to $\mathbb{C}$ we can define their sum and product in the obvious 'point-wise' way, *i.e.* as:

$$\begin{aligned}(\xi+\zeta)(g) &= \xi(g)+\zeta(g)\\ (\xi\zeta)(g) &= \xi(g)\zeta(g)\end{aligned}$$

Proposition 2.1.8. *Let $\rho_V : G \to GL(V)$ and $\rho_W : G \to GL(W)$ be representations, and let χ_V and χ_W be their characters.*

(i) $\chi_{V\oplus W} = \chi_V + \chi_W$

(ii) $\chi_{V\otimes W} = \chi_V \chi_W$

(iii) $\chi_{V^*} = \overline{\chi_V}$

(iv) $\chi_{\mathrm{Hom}(V,W)} = \overline{\chi_V}\chi_W$

Formula (ii) is particularly useful, it means we can now forget the definition of the tensor product for most purposes.

Proof. (i) Pick bases for V and W, and pick $g \in G$. Suppose that $\rho_V(g)$ and $\rho_W(g)$ are described by matrices M and N in these bases. Then $\rho_{V\oplus W}(g)$ is described by the block-diagonal matrix

$$\begin{pmatrix} M & 0 \\ 0 & N \end{pmatrix}$$

So

$$\mathrm{Tr}\left(\rho_{V\oplus W}(g)\right) = \mathrm{Tr}(M) + \mathrm{Tr}(N) = \mathrm{Tr}\left(\rho_V(g)\right) + Tr\left(\rho_W(g)\right)$$

(ii)$\rho_{V\otimes W}(g)$ is given by the matrix

$$[M\otimes N]_{js,it} = M_{ji}N_{st}$$

The trace of this matrix is

$$\begin{aligned}\sum_{i,t}[M\otimes N]_{it,it} &= \sum_{i,t} M_{ii}N_{tt}\\ &= \mathrm{Tr}(M)\,\mathrm{Tr}(N)\end{aligned}$$

i.e. $\chi_{V\otimes W}(g) = \chi_V(g)\chi_W(g)$.
(iii) $\rho_{V^*}(g)$ is described by the matrix M^{-T}, so

$$\begin{aligned}\mathrm{Tr}\left(\rho_{V^*}(g)\right) &= \mathrm{Tr}(M^{-T})\\ &= \mathrm{Tr}(M^{-1})\\ &= \chi_V(g^{-1})\\ &= \overline{\chi}_V(g) \quad \text{(by Proposition 2.1.5(ii))}\end{aligned}$$

i.e. $\chi_{V^*}(g) = \overline{\chi_V(g)}$.
(iv) By Corollary 1.10.6, the representation $\mathrm{Hom}(V,W)$ is isomorphic to the representation $V^*\otimes W$, so the statement follows by parts (ii) and (iii). □

If ρ is an irreducible representation, we say that χ_ρ is an **irreducible character**. We know that any group G has a finite list of irreps

$$U_1, \ldots, U_r$$

so there is a corresponding list of irreducible characters

$$\chi_1, \dots, \chi_r$$

We also know that any representation is a direct sum of copies of these irreps, *i.e.* if $\rho : G \to GL(V)$ is a representation then there exist numbers $m_1, \dots, m_r$ such that

$$V = U_1^{\oplus m_1} \oplus \dots \oplus U_r^{\oplus m_r}$$

Then by Proposition 2.1.8(i) we have:

$$\chi_\rho = m_1\chi_1 + \dots + m_r\chi_r$$

So every character is a linear combination of the irreducible characters, with coefficients in $\mathbb{N}$. For example, using Theorem 1.7.8 we see that the regular character χ_{reg} must satisfy

$$\chi_{reg} = d_1\chi_1 + \dots + d_r\chi_r$$

where $d_1, ..., d_r$ are the dimensions of the irreps of G.

Example 2.1.9. Let $G = C_k = \langle g \mid g^k = e\rangle$. The irreps of G are the 1-dimensional representations

$$\rho_q \colon\ g \mapsto e^{\frac{2\pi i}{k}q}, \qquad q \in [0, k-1]$$

So the irreducible characters are the functions

$$\chi_q = \rho_q \colon\ G \to \mathbb{C}$$

(remember that for 1-dimensional representations the character is the same thing as the representation). So the regular character χ_{reg} must be the sum of these:

$$\chi_{reg} = \chi_0 + \dots + \chi_{k-1}$$

We can check this against Lemma 2.1.7. We have:

$$\begin{aligned}\chi_{reg}(e) &= \chi_0(e) + \dots + \chi_{k-1}(e)\\ &= 1 + \dots + 1 = k = |G|\end{aligned}$$

and

$$\chi_{reg}(g) = 1 + e^{\frac{2\pi i}{k}} + \dots + e^{\frac{2\pi i}{k}(k-1)} = 0$$

which is a familiar identity for roots of unity. In fact for all $s \in [1, k-1]$ the identity

$$\sum_{q=0}^{k-1} \left(e^{\frac{2\pi i}{k}}\right)^{sq} = 0$$

must hold, because both sides equal $\chi_{reg}(g^s)$.

2.2. Inner products of characters

Let $\mathbb{C}^G$ denote the set of all functions from G to $\mathbb{C}$. Then $\mathbb{C}^G$ is a vector space: we've already defined the sum of two functions, and similarly we can define scalar multiplication by

$$\begin{aligned}\lambda\xi :& G \to \mathbb{C}\\ & g \mapsto \lambda\xi(g)\end{aligned}$$

for $\xi \in \mathbb{C}^G$ and $\lambda \in \mathbb{C}$.

The space $\mathbb{C}^G$ has a basis given by the set of 'characteristic functions'

$$\{\delta_g \mid g \in G\}$$

defined by

$$\delta_g : h \to \begin{cases} 1 & \text{if} \quad h = g \\ 0 & \text{if} \quad h \neq g \end{cases}$$

To see that this indeed a basis, notice that we can write any function $\xi \in \mathbb{C}^G$ as a linear combination

$$\xi = \sum_{g \in G} \xi(g)\delta_g$$

This is an equality because both sides define the same function from G to $\mathbb{C}$, furthermore it should be obvious that this is the unique way to write ξ as a linear combination of the δ_g. Consequently, the dimension of $\mathbb{C}^G$ is the size of G.

We've seen this vector space before. Recall that the vector space V_{reg} on which the regular representation acts is, by definition, a $|G|$-dimensional vector space with a basis $\{b_g \mid g \in G\}$. Then the natural bijection of sets

$$\{\delta_g \mid g \in G\} \leftrightarrow \{b_g \mid g \in G\}$$

induces a natural isomorphism of vector spaces

$$\mathbb{C}^G \cong V_{reg}$$

Despite this we're going to keep two different notations, because we're going to think of these two vector spaces differently, and do different things with them. In particular when we we write $\mathbb{C}^g$ we'll generally ignore the fact that it carries a representation of G.

Viewing the vector space $\mathbb{C}^G$ as a space of functions, we can see an important extra structure, it carries a Hermitian inner product.

Definition 2.2.1. Let $\zeta, \xi \in \mathbb{C}^G$. We define their **inner product** by

$$\langle \xi | \zeta \rangle = \frac{1}{|G|} \sum_{g \in G} \xi(g)\overline{\zeta(g)}$$

It's easy to see that $\langle \xi | \zeta \rangle$ is linear in the first variable, and conjugate-linear in the second variable, *i.e.*

$$\langle \xi | \lambda\zeta \rangle = \overline{\lambda} \langle \xi | \zeta \rangle$$

It's also clear that

$$\langle \xi | \zeta \rangle = \overline{\langle \zeta | \xi \rangle}$$

Finally, the inner-product of any vector χ with itself is

$$\langle \xi | \xi \rangle = \sum_{g \in G} |\xi(g)|^2$$

which is a non-negative real number, and equal to zero iff $\xi = 0$. The (positive) square-root of this number is called the *norm* of ξ. This list of properties are the definition of a Hermitian inner product.

You should recall that there is a *standard inner product* on the vector space $\mathbb{C}^n$, defined by

$$\langle x | y \rangle = x_1 \bar{y_i} + \ldots + x_n \bar{y_n}$$

and often written as '$x.y$'. This is an example of a Hermitian inner product. If we identify $\mathbb{C}^G$ with $\mathbb{C}^n$ (where $n = |G|$) using our basis $\{\delta_g\}$, then our inner product is almost the same as the standard one, they only differ by the overall scale factor $\frac{1}{|G|}$.

The standard basis $e_1, .., e_n \in \mathbb{C}^n$ are *orthonormal* with respect to the standard inner product, which means that $e_i.e_j = 0$ if $i \neq j$ (they're *orthogonal*), and $e_i.e_i = 1$ (they each have norm 1). Notice that our basis elements $\{\delta_g\}$ for $\mathbb{C}^G$ are not quite orthonormal with respect to the inner product that we've defined, because we have

$$\langle \delta_g | \delta_h \rangle = \begin{cases} 0 & \text{if} \quad h \neq g \\ \frac{1}{|G|} & \text{if} \quad h = g \end{cases}$$

Recall that the characters of G are elements of $\mathbb{C}^G$, so we can evaluate this inner product on pairs of characters. The answer turns out to be very useful.

Theorem 2.2.2. *Let (V, ρ_V) and (W, ρ_W) be representations of G, and let χ_V, χ_W be their characters. Then*

$$\langle \chi_W | \chi_V \rangle = \dim \mathrm{Hom}_G(V, W)$$

In particular, the inner product of two characters is always a non-negative integer. This is a strong restriction, because the inner product of two arbitrary functions (*i.e.* not necessarily characters) can be any complex number.

Before we begin the proof, two quick lemmas:

Lemma 2.2.3. *Let V be a vector space (of dimension n), and let f_1, f_2 be linear maps from V to V. Then for any scalars $\lambda_1, \lambda_2 \in \mathbb{C}$, we have*

$$\mathrm{Tr}(\lambda_1 f_1 + \lambda_2 f_2) = \lambda_1 \mathrm{Tr}(f_1) + \lambda_2 \mathrm{Tr}(f_2)$$

In other words, Tr is a linear map

$$\mathrm{Tr} : \mathrm{Hom}(V, V) \to \mathbb{C}$$

Proof. Pick any basis for V, and let M_1 and M_2 be the matrices describing f_1 and f_2 in this basis. Then

$$\begin{aligned} \mathrm{Tr}(\lambda_1 f_1 + \lambda_2 f_2) = \mathrm{Tr}(\lambda_1 M_1 + \lambda_2 M_2) \qquad & = \sum_{i=1}^{n} (\lambda_1 M_1 + \lambda_2 M_2)_{ii} \\ = \lambda_1 \sum_{i=1}^{n} (M_1)_{ii} + \lambda_2 \sum_{i=1}^{n} (M_2)_{ii} \qquad & = \lambda_1 \mathrm{Tr}(f_1) + \lambda_2 \mathrm{Tr}(f_2) \end{aligned}$$

□

Lemma 2.2.4. *Let V be a vector space, with subspace $U \subset V$, and let $\pi : V \to V$ a projection onto U. Then*

$$\mathrm{Tr}(\pi) = \dim U$$

Proof. Recall that $V = U \oplus \mathrm{Ker}(\pi)$. Pick bases for U and $\mathrm{Ker}(\pi)$, so together they form a basis for V. In this basis, π is given by the block-diagonal matrix

$$\begin{pmatrix} \mathbf{1} & 0 \\ 0 & 0 \end{pmatrix}$$

with $\dim U$ being the size of the top block and $\dim \mathrm{Ker}(\pi)$ being the size of the bottom block. So $\mathrm{Tr}(\pi) = \mathrm{Tr}(\mathbf{1}_U) = \dim U$. □

Now we can present the proof of the Theorem.

Proof of Theorem 2.2.2. Recall that

$$\mathrm{Hom}_G(V, W) \subset \mathrm{Hom}(V, W)$$

is the invariant subrepresentation, and that the map

$$\Psi : \mathrm{Hom}(V,W) \to \mathrm{Hom}(V,W)$$
$$f \mapsto \frac{1}{|G|}\sum_{g\in G} \rho_{Hom}(g)(f)$$

is a projection onto $\mathrm{Hom}_G(V,W)$ (see Proposition 1.9.12). We claim that

$$\mathrm{Tr}(\Psi) = \langle \chi_W | \chi_V \rangle$$

By Lemma 2.2.4, this would prove the theorem.
Our projection map Ψ is a linear combination of the maps

$$\rho_{\mathrm{Hom}(V,W)}(g) : \mathrm{Hom}(V,W) \to \mathrm{Hom}(V,W)$$

Therefore, by Lemma 2.2.3 we have

$$\mathrm{Tr}(\Psi) = \frac{1}{|G|}\sum_{g\in G} \mathrm{Tr}(\rho_{\mathrm{Hom}(V,W)}(g))$$

However,

$$\mathrm{Tr}(\rho_{\mathrm{Hom}(V,W)}(g)) = \chi_{\mathrm{Hom}(V,W)}(g)$$

by definition, and

$$\chi_{\mathrm{Hom}(V,W)}(g) = \overline{\chi_V(g)}\chi_W(g)$$

by Proposition 2.1.8(iv). Therefore

$$\mathrm{Tr}(\Psi) = \frac{1}{|G|}\sum_{g\in G} \overline{\chi_V(g)}\chi_W(g) = \langle \chi_W | \chi_V \rangle$$

□

Corollary 2.2.5. *Let $\chi_1, \ldots, \chi_r$ be the irreducible characters of G. Then*

$$\langle \chi_i \,|\, \chi_j \rangle = \begin{cases} 1 & \textit{if} \quad i = j \\ 0 & \textit{if} \quad i \neq j \end{cases}$$

Proof. Let χ_i and χ_j be the characters of the irreps U_i, U_j. Then

$$\langle \chi_i | \chi_j \rangle = \dim \mathrm{Hom}_G(U_i, U_j) = \begin{cases} 1 & \text{if} \quad U_i, U_j \text{ isomorphic} \\ 0 & \text{if} \quad U_i, U_j \text{ not isomorphic} \end{cases}$$

by Schur's Lemma. □

So the irreducible characters form a set of orthonormal vectors in $\mathbb{C}^G$. Therefore, if we take any linear combination of them

$$\xi = \lambda_1\chi_1 + \ldots \lambda_r\chi_r \in \mathbb{C}^G$$

(with $\lambda_1, \ldots, \lambda_r \in \mathbb{C}$) then we can calculate the coefficient of χ_i in ξ as the inner product

$$\langle \xi | \chi_i \rangle = \lambda_i$$

In particular the irreducible characters must be linearly independent, because if we have some co-efficients such that

$$\lambda_1\chi_1 + \ldots \lambda_r\chi_r = 0$$

then the above formula tells us that each λ_i is equal to zero.
Now take a representation $\rho : G \to GL(V)$, and look at its character χ_ρ. We know that χ_ρ can be written as a linear combination

$$\chi_\rho = m_1\chi_1 + \ldots + m_r\chi_r$$

of the irreducible characters, because the representation V can be decomposed into irreps. The m_i are non-negative integers, they count the number of copies of the irrep U_i occuring in V. We can calculate these coefficients by calculating the inner product

$$\langle \chi_\rho | \chi_i \rangle = m_i$$

We've just proved:

Corollary 2.2.6. *Let (V, ρ) be a representation and let χ_ρ be its character. Then the number of copies of the irrep U_i occuring in the irrep decomposition of V is given by the inner product $\langle \chi_\rho | \chi_i \rangle$.*

We can view this as a combination of Theorem 2.2.2 and Proposition 1.7.5, because

$$\langle \chi_\rho | \chi_i \rangle = \dim \mathrm{Hom}_G(U_i, V)$$

This gives us an extremely efficient way to calculate irrep decompositions.

Example 2.2.7. Let $G = C_4 = \langle g \mid g^4 = e \rangle$. Here's a 2-dimensional representation:

$$\begin{aligned} \rho : & g \mapsto M = \begin{pmatrix} i & 2 \\ 1 & -i \end{pmatrix} \\ & g^2 \mapsto M^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ & g^3 \mapsto M^3 = M \end{aligned}$$

The character of ρ takes values

	e	g	g^2	g^3
χ_ρ	2	0	2	0

The irreducible characters of G are χ_q, $q \in [0, 3]$, given by

	e	g	g^2	g^3
χ_q	1	i^q	i^{2q}	i^{3q}

(these are the irreps ρ_q). So

$$\begin{aligned} \langle \chi_\rho | \chi_q \rangle &= \frac{1}{4}\left(2 \times \overline{1} + 0 \times \overline{(i^q)} + 2 \times \overline{(i^{2q})} + 0 \times \overline{(i^{3q})}\right) \\ &= \frac{1}{2}\left(1 + (-1)^q\right) \\ &= \begin{cases} 1 & \text{if } \quad q = 0, 2 \\ 0 & \text{if } \quad q = 1, 3 \end{cases} \end{aligned}$$

So ρ is the direct sum of ρ_0 and ρ_2.

Here's another Corollary of Theorem 2.2.2.

Corollary 2.2.8. *Let χ be a character of G. Then χ is irreducible if and only if*

$$\langle \chi | \chi \rangle = 1$$

Proof. Write χ as a linear combination

$$\chi = m_1 \chi_1 + \ldots + m_r \chi_r$$

of the irreducible characters, for some non-negative integers $m_1, \ldots, m_r$. Then

$$\begin{aligned} \langle \chi | \chi \rangle &= \sum_{i,j \in [1,r]} m_i m_j \langle \chi_i | \chi_j \rangle \\ &= m_1^2 + \ldots + m_r^2 \end{aligned}$$

by Corollary 2.2.5. So $\langle \chi | \chi \rangle = 1$ iff exactly one of the $m_i = 1$ and the rest are 0. □

Recall (Proposition 2.1.5(i)) that a character gives the same value on conjugate elements of G. For $g \in G$, we write $[g]$ for the set of elements of G that are conjugate to g, this is called the **conjugacy class** of g. If we want to calculate the inner product of two characters χ_V, χ_W, we don't need to evaluate $\chi_V(g)\overline{\chi_W}(g)$ on each group element, we just need to evaluate it once on each conjugacy class, *i.e.*

$$\begin{aligned}\langle \chi_V | \chi_W \rangle &= \frac{1}{|G|} \sum_{g \in G} \chi_V(g)\overline{\chi_W}(g) \\ &= \frac{1}{|G|} \sum_{[g]} |[g]| \chi_V(g)\overline{\chi_W}(g)\end{aligned}$$

Example 2.2.9. Let $G = D_4 = \langle g, h \mid g^4 = h^2 = e,\ hgh = g^{-1} \rangle$. The conjugacy classes in G are (see Problem Sheets):

conjugacy class	$[e]$	$[g]$	$[h]$	$[gh]$	$[g^2]$
size	1	2	2	2	1

Here is a two-dimensional representation of G:

$$\rho(g) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \rho(h) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

(this is just the 'square' representation from Example 1.3.7 written in a particular basis). Then the character χ of this representation takes values

cc	$[e]$	$[g]$	$[h]$	$[gh]$	$[g^2]$
size	1	2	2	2	1
χ	2	0	0	0	-2

We compute

$$\langle \chi | \chi \rangle = \frac{1}{8}\left((1 \times 2^2) + 0 + 0 + 0 + (1 \times (-2)^2) \right) = 1$$

so ρ must be irreducible.

Now we can prove, as promised, that the character completely determines the representation. It's really just another corollary of Theorem 2.2.2.

Theorem 2.2.10. *Let $\rho_V \colon G \to GL(V)$ and $\rho_W \colon G \to GL(W)$ be representations, and suppose that $\chi_V = \chi_W$. Then V and W are isomorphic.*

This may look surprising at first, but it's just a consequence of the fact that there aren't very many representations of G.

Proof. Let $U_1, \ldots, U_r$ be the irreps of G, and $\chi_1, \ldots, \chi_r$ be their characters. We have

$$V = U_1^{\oplus m_1} \oplus \ldots \oplus U_r^{\oplus m_r}$$

for some numbers $m_1, \ldots, m_r$, and

$$W = U_1^{\oplus l_1} \oplus \ldots \oplus U_r^{\oplus l_r}$$

for some numbers $l_1, \ldots, l_r$. So

$$\chi_V = m_1\chi_1 + \ldots + m_r\chi_r$$

and

$$\chi_W = l_1\chi_1 + \ldots + l_r\chi_r$$

Since $\chi_V = \chi_W$, we have
$$m_i = \langle \chi_V | \chi_i \rangle = \langle \chi_W | \chi_i \rangle = l_i$$
for all i. So V and W have the same irrep decompositions, and hence they're isomorphic. $\square$

So we can understand everything about representations in terms of characters, and characters are much easier to work with.

Example 2.2.11. Let $G = D_4$. We know (Example 1.7.13) that G has 4 one-dimensional irreps U_1, U_2, U_3, U_4, and 1 two-dimensional irrep U_5 (the 'square' representation). So a complete list of the irreducible characters of D_4 is

cc	$[e]$	$[g]$	$[h]$	$[gh]$	$[g^2]$
size	1	2	2	2	1
χ_1	1	1	1	1	1
χ_2	1	-1	1	-1	1
χ_3	1	1	-1	-1	1
χ_4	1	-1	-1	1	1
χ_5	2	0	0	0	-2

This is called a **character table**. We can quickly see the following facts:

- U_5 is the only faithful irrep (using Corollary 2.1.6).
- For each i, we have $\overline{\chi_i} = \chi_i$ since they only take real values. Therefore each U_i is isomorphic to its own dual $(U_i)^*$.
- If $1 \leq i \leq 4$, then we have $\chi_i \chi_5 = \chi_5$. Therefore $U_i \otimes U_5$ is isomorphic to U_5.
- Now let's compute $U_5 \otimes U_5$. We have
$$\chi_5 \chi_5 = \chi_1 + \chi_2 + \chi_3 + \chi_4$$
(in general we could find the coefficients on the right-hand-side by using the inner product, but in this case it's quicker to just spot the answer). So $U_5 \otimes U_5$ is isomorphic to $U_1 \oplus U_2 \oplus U_3 \oplus U_4$.

Example 2.2.12. Let $G = D_4$ again. Suppose we'd found all the 1-dimensional irreps of G, and we'd deduced that there must a be a 2-dimensional irrep, but we couldn't work out what it was (perhaps we were just algebraists and didn't know any geometry). Then we could write down most of the character table, but the bottom line would read
$$\chi_5 \,\big|\, 2 \quad a \quad b \quad c \quad d$$
where a, b, c, d are unknown. Corollary 2.2.5 lets us deduce the unknown values, without finding the representation U_5. For each $1 \leq i \leq 4$, we must have $\langle \chi_i | \chi_5 \rangle = 0$. This gives the four equations
$$\begin{aligned} 2 + 2(a + b + c) + d &= 0 \\ 2 + 2(-a + b - c) + d &= 0 \\ 2 + 2(a - b - c) + d &= 0 \\ 2 + 2(-a - b + c) + d &= 0 \end{aligned}$$
Comparing the first equation with the remaining three, we get
$$a + c = b + c = a + b = 0$$
so $a = b = c = 0$, and then $d = -2$.

There's another trick we could have used here which makes the calculation even easier. Consider the product:

$$\chi_2\chi_5 \,\big|\, 2 \quad -a \quad b \quad -c \quad d$$

This is the character of $U_2 \otimes U_5$, so by Claim 1.10.2 it must be an irreducible character, and the only possiblity is that $\chi_2\chi_5 = \chi_5$. This immediately implies $a = c = 0$, and by considering $\chi_3\chi_5$ instead we get $b = 0$ too. Then to find d we use one of the equations from above.

Characters are also very helpful for understanding induced representations. In Section 1.8 we saw that, given a subgroup H and a representation V of H, we can construct a representation $\mathrm{Ind}(V)$ of G. As with tensor products, it's much quicker to study the character of $\mathrm{Ind}(V)$ instead of computing all the matrices. The key is to use Frobenius reciprocity.

Proposition 2.2.13 (Frobenius reciprocity, v2)**.** *Let $H \subset G$ be a subgroup. Let V be a representation of H, let W be a representation of G and let $\chi_V : H \to \mathbb{C}$ and $\chi_W : G \to \mathbb{C}$ be their characters. Let $\chi_{\mathrm{Ind}(V)} : G \to \mathbb{C}$ be the character of the induced represention* $\mathrm{Ind}(V)$*. Then:*

$$\langle \chi_{\mathrm{Ind}(V)} \,|\, \chi_W \rangle = \langle \chi_V \,|\, \chi_W \rangle$$

On the left of this equation we're using the inner product in $\mathbb{C}^G$, and on the right the inner product in $\mathbb{C}^H$. Note that χ_W does define an element of $\mathbb{C}^H$, just by restricting it to the subset H (and this is obviously the character of the restricted representation).

Proof. This follows immediately from Theorem 1.8.8 and Theorem 2.2.2. □

Example 2.2.14. Let $G = D_4$ and let:

$$H = \{e, h\} \subset D_4$$

This is a subgroup of index 4. If V is the trivial irrep of H then $\mathrm{Ind}(V)$ is a 4-dimensional representation of G (we also used the notation ρ^H for this representation). We can compute $\mathrm{Ind}(V)$ by computing its character.

Let $\chi_1, ..., \chi_5$ denote the irreducible characters of D_4, as in Example 2.2.11. Since $H = C_2$ it has two irreducible characters $\tilde{\chi}_1$ (trivial) and $\tilde{\chi}_2$ (sign). If we restrict any χ_i to H we must get some character of C_2, and from the table we can see immediately that

$$\chi_1|_H = \chi_2|_H = \tilde{\chi}_1, \qquad \chi_3|_H = \chi_4|_H = \tilde{\chi}_2$$

and:

$$\chi_5|_H = \tilde{\chi}_1 + \tilde{\chi}_2$$

So if we restrict U_5 to H it splits as the direct sum of the trivial and sign irreps. Since $\chi_V = \tilde{\chi}_1$, Frobenius reciprocity tells us that

$$\langle \chi_{\mathrm{Ind}(V)} \,|\, \chi_i \rangle = \begin{cases} 1 & \text{if } i = 1, 2, 5 \\ 0 & \text{if } i = 3, 4. \end{cases}$$

So

$$\chi_{\mathrm{Ind}(V)} = \chi_1 + \chi_2 + \chi_5$$

and hence:

$$\mathrm{Ind}(V) = U_1 \oplus U_2 \oplus U_5$$

If instead we'd set V to be the sign representation of H, the induced representation would be $U_3 \oplus U_4 \oplus U_5$.

2.3. Class functions and character tables

Definition 2.3.1. A **class function** for a group G is a function

$$\xi\colon G \to \mathbb{C}$$

such that

$$\xi(h^{-1}gh) = \xi(g)$$

for all $g, h \in G$.

So a class function is a function in $\mathbb{C}^G$ that is constant on each conjugacy class. It's elementary to check that the class functions form a subspace of $\mathbb{C}^G$, we denote it by

$$\mathbb{C}^G_{cl} \subset \mathbb{C}^G$$

Recall that $\mathbb{C}^G$ has a basis given by the functions

$$\delta_g\colon h \mapsto \begin{cases} 1 & \text{if } h = g \\ 0 & \text{if } h \neq g \end{cases}$$

Similarly, $\mathbb{C}_{cl}$ has a basis given by the functions

$$\delta_{[g]} = \sum_{\tilde{g}\in[g]} \delta_{\tilde{g}}\colon h \mapsto \begin{cases} 1 & \text{if } h \in [g] \\ 0 & \text{if } h \notin [g] \end{cases}$$

So $\dim(\mathbb{C}^G_{cl})$ is the number of conjugacy classes in G.
The space $\mathbb{C}^G_{cl}$ carries a Hermitian inner product, it's just given by the restriction of the inner product from $\mathbb{C}^G$. Notice that if we want to calculate $\langle \xi|\zeta\rangle$ where ξ and ζ are class functions, then we only need to evaluate $\xi(g)\overline{\zeta}(g)$ once on each conjugacy class, so the formula becomes

$$\langle \xi|\zeta\rangle = \frac{1}{|G|}\sum_{[g]} |[g]|\xi(g)\overline{\zeta}(g)$$

(we already observed this for the case when ξ and ζ are characters). Also notice that the basis $\{\delta_{[g]}\}$ is orthogonal with respect to this inner product, but not orthonormal, because we have

$$\langle \delta_{[g]}|\delta_{[h]}\rangle = \begin{cases} \frac{|[g]|}{|G|} & \text{if} \quad h = g \\ 0 & \text{if} \quad h \neq g \end{cases}$$

The numbers $\frac{|[g]|}{|G|}$ have another interpretation. Recall that the **centraliser** of a group element $g \in G$ is the subgroup

$$C_g = \{h \in G \mid hgh^{-1} = g\}$$

of elements that commute with g. By the Orbit-Stabiliser theorem,

$$|C_g| = \frac{|G|}{|[g]|}$$

so the norm of $\delta_{[g]}$ is $|C_g|^{-1}$.
Now let $\chi_1, \ldots, \chi_r$ be the irreducible characters of G. Then each χ_i is a class function, and they form an orthonormal set of vectors in $\mathbb{C}^G_{cl}$ (by Corollary 2.2.5), and hence they're linearly independent. The maximum size of a linearly independent set in a vector space is the dimension of the vector space, so

$$r \leq \dim \mathbb{C}^G_{cl}$$

We've just proved:

Proposition 2.3.2. *For any group G, the number of irreps of G is at most the number of conjugacy classes in G.*

In fact, a stronger result holds:

Theorem 2.3.3. *For any group G, the number of irreps of G equals the number of conjugacy classes in G.*

We're not going to prove this yet, we'll prove it in Section 3 once we've introduced the technology of **group algebras**.

Example 2.3.4. Let $G = S_4$. You should recall that two permutations are conjugate in S_n if and only if they have the same cycle type. So in S_4 we have conjugacy classes

$$[(1)], [(12)], [(123)], [(1234)], [(12)(34)]$$

So S_4 has 5 irreps. This agrees with what we found in Example 1.7.12.

Example 2.3.5. Let $G = S_5$. The conjugacy classes in G are

$$[(1)], [(12)], [(123)], [(1234)], [(12345)], [(12)(34)], [(123)(45)]$$

So S_5 has 7 irreps.

Theorem 2.3.3 is an unusual result. Since the number of irreps equals the number of conjugacy classes, it would be reasonable to guess that there's some natural way to pair up each irrep with a conjugacy class, and vice versa. But this is not true! There's no sensible way to pair them up. The correct way to think about it is expressed in the following corollary:

Corollary 2.3.6. *The irreducible characters $\chi_1, \ldots, \chi_r$ form a basis of $\mathbb{C}^G_{cl}$.*

Proof. The irreducible characters form a linearly independent set in $\mathbb{C}^G_{cl}$, and:

$$r = \#\{\text{conjugacy classes}\} = \dim \mathbb{C}^G_{cl}$$

$\square$

So we have two sensible bases for the vector space $\mathbb{C}^G_{cl}$, we have $\{\delta_{[g_j]}\}$ which is indexed by conjugacy classes, and $\{\chi_i\}$ which is indexed by irreps. The two bases must have the same size, but there needn't be a sensible way to pair up their elements.

We've met character tables already in Section 2.2, but let's give a formal definition:

Definition 2.3.7. Let G be a group, let $\chi_1, \ldots, \chi_r$ be the irreducible characters of G, and let $[g_1], \ldots, [g_s]$ be the conjugacy classes in G. The **character table** of G is the matrix C with entries

$$C_{ij} = \chi_i(g_j)$$

From Theorem 2.3.3 we know that $r = s$, so C is a square matrix.

Example 2.3.8. Let $G = S_3$. Let

$$g_1 = (1) \quad g_2 = (12) \quad g_3 = (123)$$

and let χ_1, χ_2, χ_3 be the characters of the trivial, sign and triangular irreps. The character table of G is the matrix

	(1)	(12)	(123)
χ_1	1	1	1
χ_2	1	-1	1
χ_3	2	0	-1

As we observed in Corollary 2.3.6, the vector space $\mathbb{C}^G_{cl}$ has two natural bases, given by the sets $\{\delta_{[g_j]}\}$ and $\{\chi_i\}$. Suppose we had some vector written in terms of the second basis, and we want to express it in terms of the first basis. Then what we need is the change-of-basis matrix between the two bases, *i.e.* we need to express each vector χ_i in terms of the basis $\{\delta_{[g_j]}\}$. But this is easy, because

$$\chi_i = \chi_i(g_1)\delta_{[g_1]} + \ldots + \chi_i(g_r)\delta_{[g_r]}$$

since both sides give the same function $G \to \mathbb{C}$. These coefficients are precisely the entries in C, *i.e.* C (or maybe $C^\top$) is the change-of-basis matrix between our two bases of $\mathbb{C}^G_{cl}$.

Let's calculate $C\overline{C}^\top$. We have

$$\begin{aligned}(C\overline{C}^\top)_{ik} &= \sum_{j=1}^r C_{ij}\overline{C}_{kj} \\ &= \sum_{j=1}^r \chi_i(g_i)\overline{\chi_k}(g_j)\end{aligned}$$

This looks like the formula for $\langle\chi_i|\chi_j\rangle$ but it's missing the coefficients $\frac{|[g_j]|}{|G|}$. If we modify C by replacing it with the matrix

$$B_{ij} = \chi_i(g_j)\sqrt{\frac{|[g_j]|}{|G|}}$$

Then

$$\begin{aligned}(B\overline{B}^\top)_{ik} &= \sum_{i=1}^r \chi_i(g_j)\bar{\chi_k}(g_j)\frac{|[g_j]|}{|G|} \\ &= \langle\chi_i|\chi_k\rangle \\ &= \begin{cases} 1 & \text{if } i = k \\ 0 & \text{if } i \neq k \end{cases}\end{aligned}$$

So $B\overline{B}^\top = \mathbf{1}_r$, *i.e.* B is a unitary matrix.

Recall that a unitary matrix is exactly a change-of-basis matrix between two orthonormal bases of a complex vector space (it's the analogue of an orthogonal matrix for a real vector space). This explains why we have to make this modification, the problem is that the basis $\{\delta_{[g_j]}\}$ is not orthonormal. The elements are orthogonal, but they have norms $\frac{|[g_i]|}{|G|}$. However if we rescale each element we get an orthonormal basis

$$\left\{ \delta_{[g_j]}\sqrt{\frac{|G|}{|[g_j]|}} \right\} \subset \mathbb{C}^{cl}$$

and B is the change-of-basis matrix between this basis and the orthonormal basis $\{\chi_i\}$. This is why B is a unitary matrix, and C is not.

Example 2.3.9. In Example 2.3.8 we wrote down the character table C for the group S_3. The three conjugacy classes have sizes 1,3 and 2, and dividing by $|G| = 6$ gives $\frac{1}{6}, \frac{1}{2}, \frac{1}{3}$. So the modified character table is

$$B = \begin{pmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{2}{\sqrt{6}} & 0 & -\frac{1}{\sqrt{3}} \end{pmatrix}$$

which is a unitary matrix.

Proposition 2.3.10. *Let $[g_i]$ and $[g_j]$ be two conjugacy classes in G. Then*

$$\sum_{k=1}^{r} \overline{\chi_k}(g_i)\chi_k(g_j) = \begin{cases} \frac{|G|}{|[g_i]|} & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

Proof. Let B be the modified character table. We have $B^{-1} = \overline{B}^T$, so $\overline{B}^T B = \mathbf{1}_r$, *i.e.*

$$\begin{aligned} \sum_{k=1}^{r} \overline{B_{ki}} B_{kj} &= \left(\sum_{k=1}^{r} \overline{\chi_k}(g_i)\chi_k(g_i) \right) \frac{\sqrt{|[g_i]||[g_j]|}}{|G|} \\ &= \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \end{aligned}$$

which implies the proposition. □

Remember that $\frac{|G|}{|[g_i]|} = |C_{g_i}|$, the size of the centralizer of g_i, so this is always a positive integer.

Now we have two useful sets of equations on the character table C:

- **Row Orthogonality**:

$$\sum_{j=1}^{r} C_{ij}\overline{C_{kj}}|[g_j]| = \begin{cases} |G| & \text{if } i = k \\ 0 & \text{if } i \neq k \end{cases}$$

- **Column Orthogonality**:

$$\sum_{k=1}^{r} \overline{C_{ki}} C_{kj} = \begin{cases} \frac{|G|}{|[g_i]|} = |C_{g_i}| & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

Both of them are really the statement that the modified character table B is unitary.

Example 2.3.11. Let's solve Example 2.2.12 again using column orthogonality. We have a partial character table

cc	$[e]$	$[g]$	$[h]$	$[gh]$	$[g^2]$
size	1	2	2	2	1
$\lvert G\rvert$/size	8	4	4	4	8
χ_1	1	1	1	1	1
χ_2	1	-1	1	-1	1
χ_3	1	1	-1	-1	1
χ_4	1	-1	-1	1	1
χ_5	2	a	b	c	d

The $[g]$-column gives the equation

$$1 + 1 + 1 + 1 + |a|^2 = 4$$

so $a = 0$, and similarly $b = c = 0$. Orthogonality of the $[e]$-column and the $[g^2]$-column gives the equation

$$1 + 1 + 1 + 1 + 2d = 0$$

so $d = -2$.

The column orthogonality equations carry exactly the same information as the row orthogonality equations, but sometimes it's quicker to use one rather than the other (or we can use a mixture of both).

3. Algebras and modules

3.1. Algebras

Consider the vector space $\mathrm{Mat}_{n\times n}(\mathbb{C})$ of all $n \times n$ matrices. As well as being a vector space we have the extra structure of matrix multiplication, which is a map

$$m\colon \mathrm{Mat}_{n\times n}(\mathbb{C}) \times \mathrm{Mat}_{n\times n}(\mathbb{C}) \to \mathrm{Mat}_{n\times n}(\mathbb{C})$$
$$m\colon (M, N) \mapsto MN$$

This map has the following properties, it is:

(i) Bilinear: it's linear in each variable.

(ii) Associative:
$$m(m(L, M), N) = m(L, m(M, N))$$
i.e.
$$(LM)N = L(MN)$$

(iii) Unital: there's an element $I_n \in \mathrm{Mat}_{n\times n}(\mathbb{C})$ obeying $m(M, I) = m(I, M) = M$ for all $M \in \mathrm{Mat}_{n\times N}(\mathbb{C})$.

A structure like this is called an **algebra**.

Definition 3.1.1. An **algebra** is a vector space A equipped with a map
$$m : A \times A \to A$$
that is bilinear, associative and unital.

We'll usually write ab when we mean $m(a, b)$. Associativity means that the expression abc is well defined (without brackets). We'll generally write 1_A for the unit element.

Example 3.1.2. (i) The 1-dimensional vector space $\mathbb{C}$ is an algebra, with m the usual multiplication.

(ii) Let $A = \mathbb{C} \oplus \mathbb{C}$, with multiplication
$$m((x_1, y_1), (x_2, y_2)) = (x_1x_2, y_1y_2)$$
Then A is an algebra. The unit is $(1, 1)$.

(iii) Let $A = \mathbb{C}[x]$, the (infinite-dimensional) vector space of polynomials in x. A is an algebra under the usual multiplication of polynomials, with unit the constant polynomial $1_A = 1$. More generally, we have an algebra $A = \mathbb{C}[x_1, \ldots, x_n]$ of polynomials in n variables.

(iv) Let A be the 2-dimensional space with basis $\{1, x\}$. Define $1^2 = 1$, $1x = x1 = x$ and $x^2 = 0$, and extend bilinearly. Then A is an algebra.

(v) Let V be a vector space, and let $A = \mathrm{Hom}(V, V)$. Multiplication is given by composition of maps. If we pick a basis for V, then A becomes $\mathrm{Mat}_{n\times n}(\mathbb{C})$, and composition of maps corresponds to multiplication of matrices (Proposition A.2.3). We'll often call an algebra of this form a 'matrix algebra' even if we haven't chosen a basis for V.

Except for example (iii), in each of these cases A is a finite-dimensional space. From now on we'll assume our algebras are finite-dimensional. Examples (i)-(iv) are all **commutative**, *i.e.* $ab = ba$ for all $a, b \in A$. We will *not* assume that our algebras are commutative.

Notes on terminology:

(i) An algebra is very similar to a *ring*, but using a vector space instead of an abelian group. Also rings are normally assumed to be commutative.

(ii) If you have encountered Lie algebras you will see that, under our definition, a Lie algebra is *not* a special case of an algebra. Sometimes the name *associative algebra* is used for our definition.

For this course, the most important algebras are given by the following construction:

Let G be a (finite) group. Then we can construct an algebra from G. Suppose the elements of G are given by $\{g_1, \ldots, g_t\}$. Consider the set of formal linear combinations of elements of G:

$$\mathbb{C}[G] = \{\lambda_1 g_1 + \ldots + \lambda_t g_t \mid \lambda_1 \ldots \lambda_t \in \mathbb{C}\}$$

This set is a vector space, we define

$$(\lambda_1 g_1 + \ldots + \lambda_t g_t) + (\mu_1 g_1 + \ldots + \mu_t g_t) = (\lambda_1 + \mu_1) g_1 + \ldots + (\lambda_t + \mu_t) g_t$$

and

$$\mu(\lambda_1 g_1 + \ldots + \lambda_t g_t) = (\mu\lambda_1) g_1 + \ldots + (\mu\lambda_t) g_t$$

The set G sits inside $\mathbb{C}[G]$ as the subset where one coefficient is 1 and the rest are zero. This set forms a basis for $\mathbb{C}[G]$, in fact we could have defined $\mathbb{C}[G]$ to be the vector space with G as a basis.

As well as being a vector space $\mathbb{C}[G]$ is also an algebra, we call it the **group algebra** of G. To define the product of two basis elements we just use the group product, *i.e.* we define:

$$m(g,h) = gh \quad \in G \subset \mathbb{C}[G]$$

Then we extend this to a bilinear map

$$m : \mathbb{C}[G] \times \mathbb{C}[G] \to \mathbb{C}[G]$$

i.e. we define

$$\begin{aligned}(\lambda_1 g_1 + \ldots + \lambda_t g_t)(\mu_1 g_1 + \ldots + \mu_t g_t) &= \lambda_1\mu_1(g_1^2) + \ldots + \lambda_t\mu_1(g_t g_1) + \ldots \\ &\quad \ldots + \lambda_1\mu_t(g_1 g_t) + \ldots + \lambda_t\mu_t(g_t^2) \\ &= \sum_k \left(\sum_{\substack{i,j \text{ such that} \\ g_i g_j = g_k}} \lambda_i \mu_j g_k \right)\end{aligned}$$

This product is associative because the product in G is associative, and it has unit $e \in G \subset \mathbb{C}[G]$.

Example 3.1.3. Let $G = C_2 = \langle e, g \mid g^2 = e \rangle$. Then

$$\mathbb{C}[G] = \{\lambda_1 e + \lambda_2 g \mid \lambda_1, \lambda_2 \in \mathbb{C}\}$$

is a two-dimensional vector space, with multiplication

$$\begin{aligned}(\lambda_1 e_1 + \lambda_2 g)(\mu_1 e + \mu_2 g) &= \lambda_2\mu_1 e + \lambda_1\mu_2 g + \lambda_2\mu_1 g + \lambda_2\mu_2 e \\ &= (\lambda_1\mu_1 + \lambda_2\mu_2)e + (\lambda_1\mu_2 + \lambda_2\mu_1)g\end{aligned}$$

We've met this vector space $\mathbb{C}[G]$ before. Recall that the regular representation of G acts on a vector space V_{reg} which has a basis $\{b_g \mid g \in G\}$. So V_{reg} is naturally isomorphic to $\mathbb{C}[G]$, via $b_g \leftrightarrow g$. We've also met it as the space $\mathbb{C}^G$ of $\mathbb{C}$-valued functions on G, which has a basis $\{\delta_g \,|\, g \in G\}$.

Warning: For functions $\xi, \zeta \in \mathbb{C}^G$ we defined a 'point-wise' product

$$\xi\zeta \colon g \mapsto \xi(g)\zeta(g)$$

This makes $\mathbb{C}^G$ into an algebra, but it's completely different from the group algebra $\mathbb{C}[G]$. In the pointwise product:

$$\delta_g \delta_h = \begin{cases} \delta_g & \text{if } g = h \\ 0 & \text{if } g \neq h \end{cases}$$

In particular, $\mathbb{C}^G$ is always commutative, whereas $\mathbb{C}[G]$ is commutative iff G is abelian.

We can also define $\mathbb{C}[G]$ for an infinite group, but then we get an infinite-dimensional algebra.

Definition 3.1.4. A **homomorphism** between algebras A and B is a linear map

$$f : A \to B$$

such that

(i) $f(1_A) = 1_B$
(ii) $f(a_1 a_2) = f(a_1)f(a_2), \ \ \forall a_1, a_2 \in A$

An **isomorphism** between A and B is a homomorphism that's also an isomorphism of vector spaces.

Example 3.1.5. Let $A = \langle 1, x \rangle$ where $x^2 = 0$ (from Example 3.1.2(iv)). There's a homomorphism

$$\begin{aligned} f_0 \colon A &\to \mathbb{C} \\ 1_A &\mapsto 1 \\ x &\mapsto 0 \end{aligned}$$

In fact, f_0 is the only possible such homomorphism. If f is any homomorphism from A to $\mathbb{C}$ then we must have $f(1_A) = 1$, and

$$\begin{aligned} & f(x)f(x) = f(x^2) = f(0) = 0 \\ \Rightarrow \quad & f(x) = 0 \end{aligned}$$

Example 3.1.6. Let $A = \mathbb{C}[C_2]$ and let $B = \mathbb{C} \oplus \mathbb{C}$ as in Example 3.1.2(ii). Define

$$\begin{aligned} f \colon A &\to B \\ e &\mapsto (1, 1) \\ g &\mapsto (1, -1) \end{aligned}$$

Then f is an isomorphism of vector spaces, and it's also a homomorphism, because

$$f(1_A) = f(e) = (1, 1) = 1_B$$
$$f(g)^2 = ((1, -1))^2 = (1, 1) = f(g^2)$$

So $f(a_1, a_2) = f(a_1)f(a_2)$ for all $a_1, a_2 \in A$ by bilinearity. So A and B are isomorphic algebras.

Example 3.1.7. Let $A = \mathbb{C}[S_3]$ and let $\rho : G \to GL_2(\mathbb{C})$ be the 2-d irrep. Then we can extend ρ to a linear map

$$\tilde{\rho} : \mathbb{C}[S_3] \to \mathrm{Mat}_{2\times 2}(\mathbb{C})$$

and is a homomorphism of algebras. It cannot be an isomorphism since the dimensions are different.

Notice that the last example works in general: any representation $\rho : G \to GL(V)$ extends to an algebra homomorphism $\tilde{\rho} : \mathbb{C}[G] \to \mathrm{Hom}(V, V)$.

Now let A and B be two algebras. The vector space

$$A \oplus B$$

is naturally an algebra, it has a product

$$m\left((a_1, b_1), (a_2, b_2)\right) = (a_1 a_2, b_1 b_2)$$

The algebra axioms are easy to check, the unit is

$$1_{A\oplus B} = (1_A, 1_B)$$

We call this algebra the **direct sum** of A and B.

Notice that the direct sum of group algebras is not a group algebra in general, *i.e.* if G_1, G_2 are groups then in general there does not exist a group G_3 such that:

$$\mathbb{C}[G_1] \oplus \mathbb{C}[G_2] = \mathbb{C}[G_3]$$

So taking direct sums of algebras is a new thing, not something we can do for groups.

Example 3.1.8. If both A and B are the 1-dimensional algebra $\mathbb{C}$, then $A \oplus B = \mathbb{C} \oplus \mathbb{C}$ is the algebra we defined in Example 3.1.2(ii). If we iterate this construction we can make a k-dimensional algebra $\mathbb{C}^{\oplus k}$ for any k.

Example 3.1.9. Fix $n, m \in \mathbb{N}$ and set $d = n + m$. Consider the subspace of block diagonal matrices:

$$A = \left\{ \begin{pmatrix} M & 0 \\ 0 & N \end{pmatrix},\ M \in \mathrm{Mat}_{n\times n}(\mathbb{C}),\ N \in \mathrm{Mat}_{m\times m}(\mathbb{C}) \right\} \subset \mathrm{Mat}_{d\times d}(\mathbb{C})$$

Clearly A is an algebra under the operation of matrix multiplication, it is a subalgebra of $\mathrm{Mat}_{d\times d}(\mathbb{C})$. And we have an isomorphism of algebras:

$$A = \mathrm{Mat}_{n\times n}(\mathbb{C}) \oplus \mathrm{Mat}_{m\times m}(\mathbb{C})$$

Example 3.1.10. Let $G = S_3$ and let $\rho_1, \rho_2 : S_3 \to GL_1(\mathbb{C})$ and $\rho_3 : S_3 \to GL_2(\mathbb{C})$ be the three irreps of G. Together they define an algebra homomorphism:

$$\tilde{\rho} = (\tilde{\rho}_1, \tilde{\rho}_2, \tilde{\rho}_3) : \ \mathbb{C}[S_3] \longrightarrow A = \mathbb{C} \oplus \mathbb{C} \oplus \mathrm{Mat}_{2\times 2}(\mathbb{C})$$

As in the previous example we can view A as a subalgebra inside $\mathrm{Mat}_{4\times 4}(\mathbb{C})$, it's the subalgebra of block diagonal matrices where the blocks have sizes 1, 1 and 2. If we take this point-of-view then:

$$\tilde{\rho}(g) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \omega & 0 \\ 0 & 0 & 0 & \omega^{-1} \end{pmatrix} \quad \text{and} \quad \tilde{\rho}(h) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

So this is just the 4-dimensional representation we get by taking the direct sum of all the irreps, extended to an algebra homomorphism. Now notice that $\dim A = 6$, which is the same dimension as $\mathbb{C}[S_3]$.

Claim: $\tilde{\rho} : \mathbb{C}[S_3] \to A$ is an isomorphism of algebras.

You can check this by looking at what $\tilde{\rho}$ does to the basis elements $G \subset \mathbb{C}[S_3]$.

If you look back at Example 3.1.6 you'll see the same structure: we used all the irreps of C_2 and got an isomorphism of algebras. This process works for all groups and we'll spend most of the rest of this chapter finding out why.

3.2. Modules

Definition 3.2.1. Let A be an algebra. A (left) **A-module** is a vector space M, together with a homomorphism of algebras

$$\mu : A \to \mathrm{Hom}(M, M)$$

We'll assume all our modules are finite-dimensional (as vector spaces). Often we'll be lazy and say things like 'let M be an A-module', leaving the homomorphism μ implicit.

The definition of a module is a direct generalization of the definition of a representation of a group, in fact another name for an A-module is a 'representation of A'. For group algebras, the two concepts are the same:

Proposition 3.2.2. *Let G be a group. Then a $\mathbb{C}[G]$-module is the same thing as a representation of G.*

Proof. We already observed above that if we take a representation $\rho : G \to GL(V)$ and extend it linearly we get an algebra homomorphism $\tilde{\rho} : \mathbb{C}[G] \to \mathrm{Hom}(V, V)$. This is exactly the data of a $\mathbb{C}[G]$-module.

Now suppose that

$$\mu : \mathbb{C}[G] \to \mathrm{Hom}(M, M)$$

is any $\mathbb{C}[G]$-module. Restricting μ to the subset $G \subset \mathbb{C}[G]$ gives a function:

$$\rho = \mu|_G : G \to \mathrm{Hom}(M, M)$$

For $g, h \in G$ we have

$$\rho(gh) = \rho(g) \circ \rho(h)$$

using the fact that μ is an an algebra homomorphism, and the definition of the product in $\mathbb{C}[G]$. In particular,

$$\rho(g) \circ \rho(g^{-1}) = \rho(g^{-1}) \circ \rho(g) = \rho(e) = \mathbf{1}_M$$

since e is the unit in $\mathbb{C}[G]$. Thus each linear map $\rho(g)$ has an inverse $\rho(g^{-1})$, so ρ defines a function

$$\rho : G \to GL(M)$$

which we've just shown to be a homomorphism. So this is a representation of G. □

Example 3.2.3. Let A be the 1-dimensional algebra $\mathbb{C}$. Then an A-module is choice of vector space M and a homomorphism

$$\mu : \mathbb{C} \to \mathrm{Hom}(M, M)$$

However, we must have $\mu(1) = \mathbf{1}_M$, and by then by linearity we must have $\mu(\lambda) = \lambda \mathbf{1}_M$. Thus there is a unique such μ. So a $\mathbb{C}$-module is exactly the same thing as a vector space.

Example 3.2.4. Let $A = \langle 1, x \rangle$ with $x^2 = 0$. Suppose M is a 1-dimensional A-module, i.e. a 1-dimensional vector space M and a homomorphism:

$$\mu : A \to \mathrm{Hom}(M, M) = \mathbb{C}$$

By Example 3.1.5 there is a unique such μ, defined by $\mu(x) = 0$.

Example 3.2.5. Fix a vector space V and let A be the algebra $\mathrm{Hom}(V, V)$, as in see Example 3.1.2(v). Or if you prefer, pick a basis to get isomorphisms $V \cong \mathbb{C}^n$ and $A = \mathrm{Mat}_{n\times n}(\mathbb{C})$. The vector space V is automatically an A-module, because we can use the identity map

$$\mu = \mathbf{1} \colon\ A \to \mathrm{Hom}(V, V)$$

which is obviously an algebra homomorphism.

Example 3.2.6. $A = \mathbb{C}\oplus\mathbb{C}$, and let M be a 1-dimensional module, *i.e.* a homomorphism $\mu : A \to \mathbb{C}$. Since μ is linear it's determined by two complex numbers:

$$\mu(1,0) = z \quad \text{and} \quad \mu(0,1) = w$$

And because it's a homomorphism we must have

$$\begin{aligned} z^2 &= \mu\big((1,0)^2\big) = \mu(1,0) = z \\ w^2 &= \mu\big((0,1)^2\big) = \mu(0,1) = w \\ zw &= \mu\big((1,0).(0,1)\big) = \mu(0,0) = 0 \\ z+w &= \mu\big((1,0)+(0,1)\big) = \mu(1,1) = 1 \end{aligned}$$

There are two solutions, either $z = 1$ and $w = 0$, or vice versa. So A has exactly two 1-dimensional modules.

Recall (Example 3.1.6) that A is isomorphic to $\mathbb{C}[C_2]$, so $\mathbb{C}[C_2]$ must also have two 1-dimensional modules. But a $\mathbb{C}[C_2]$-module is the same thing a representation of C_2, and we know that there are two 1-dimensional representations of C_2. So this is consistent.

Since modules are generalizations of representations, we should be able to carry over some of the definitions and results of Section 1. Let's do this now.

We begin by generalizing the regular representation. For any algebra A, there is a canonical A-module, namely A itself. We define the module structure

$$\mu\colon A \to \mathrm{Hom}(A, A)$$

by

$$\begin{aligned} \mu(a) : A &\to A \\ b &\mapsto ab \end{aligned}$$

i.e. A acts on itself via left multiplication. The algebra axioms immediately imply that A is an A-module.

In the special case that $A = \mathbb{C}[G]$ we deduce that there is a canonical representation of G on the vector space $\mathbb{C}[G]$. This is exactly the regular representation V_{reg}.

Let's introduce some simpler notation. Suppose A is an algebra, and $\mu : A \to \mathrm{Hom}(M, M)$ is an A-module. From now on we're going to write am to mean $\mu(a)(m)$. Note that the expression abm is well-defined without brackets.

Definition 3.2.7. Let M and N be two A-modules. A **homomorphism** of A-modules, or an **A-linear map**, is a linear map

$$f\colon M \to N$$

such that

$$f(ax) = af(x)$$

for all $a \in A$ and $x \in M$.

The set of all A-linear maps from M to N is a subset

$$\mathrm{Hom}_A(M, N) \subset \mathrm{Hom}(M, N)$$

It's easy to check that it's a subspace.

Definition 3.2.8. A **submodule** of an A-module M is a subspace $N \subseteq M$ such that

$$ax \in N, \quad \forall a \in A, x \in N$$

Claim 3.2.9. *Let $A = \mathbb{C}[G]$ be a group algebra and let M and N be A-modules,* i.e. *representations of G. Then:*

(1) A submodule of M is the same thing as a subrepresentation of M.
(2) An A-linear map from M to N is the same thing as a G-linear map from M to N.

Definition 3.2.10. Let M and N be A-modules. The **direct sum** of M and N is the vector space $M \oplus N$ equipped with the A-module structure

$$a(x, y) = (ax, ay)$$

It's easy to check that $M \oplus N$ is an A-module, *i.e.* the above formula does indeed define a homorphism from A to $\mathrm{Hom}(M \oplus N, M \oplus N)$. It's also clear that this definition agrees with the definition of the direct sum of representations in the special case $A = \mathbb{C}[G]$.

Lemma 3.2.11. *Let L, M, N be three A-modules. Then there are natural isomorphisms of vector spaces*

(i) $\mathrm{Hom}_A(L, M \oplus N) = \mathrm{Hom}_A(L, M) \oplus \mathrm{Hom}_A(L, N)$
(ii) $\mathrm{Hom}_A(M \oplus N, L) = \mathrm{Hom}_A(M, L) \oplus \mathrm{Hom}_A(N, L)$

Proof. This is just the same as the proof of the corresponding result for representations, Corollary 1.7.4. □

As we've seen, many constructions from the world of representations generalize to modules over an arbitrary algebra A. However, not everything generalises. For example, the space of all linear maps $\mathrm{Hom}(M, N)$ between two A-modules is *not* in general an A-module. Similarly, the dual vector space M^* and the tensor product $M \otimes N$ are not A-modules in general (see Problem Sheets). Finally, there's no generalisation of the trivial representation. So group algebras are quite special.

Aside: If you have encountered modules before you may know that if A is commutative then it is possible to define the tensor product $M \otimes_A N$ of two A-modules, and this is an A-module. But:

(1) This doesn't work for non-commutative A.
(2) Even if G is an abelian group, so $A = \mathbb{C}[G]$ is commutative, the tensor product $M \otimes_A N$ is not the same thing as the tensor product of representations $M \otimes N$.

To end this section let's think about modules over an algebra which itself is a direct sum. Suppose we have an algebra

$$A \oplus B$$

which is the direct sum of two algebras A and B. If M is an A-module, then we can view M as an $A \oplus B$-module by defining

$$(a, b)(x) = ax$$

for $x \in M$, *i.e.* we let b act as zero on M for all $b \in B$. In other words, we take the given homomorphism $A \to \mathrm{Hom}(M, M)$ and compose it with the projection homomorphism $A \oplus B \to A$.

So any A-module M gives us an $A \oplus B$-module automatically. Similarly, every B-module N can also be viewed as an $A \oplus B$-module. So we can form the direct sum $M \oplus N$, and this is an $A \oplus B$-module. The multiplication is

$$(a, b)(x, y) = (ax, by)$$

for $x \in M$, $y \in N$.

Proposition 3.2.12. *Every module over $A \oplus B$ is isomorphic to $M \oplus N$ for some A-module M and some B-module N.*

Proof. Consider the elements $(1_A, 0)$ and $(0, 1_B)$ in the algebra $A \oplus B$. Let's denote them just by 1_A and 1_B for brevity. They satisfy the following relations:

$$1_A^2 = 1_A, \quad 1_B^2 = 1_B, \quad 1_A 1_B = 1_B 1_A = 0, \quad 1_{A\oplus B} = 1_A + 1_B$$

Now let L be any $A \oplus B$-module. So we have linear maps:

$$1_A \colon L \to L \qquad \text{and} \qquad 1_B \colon L \to L$$

Let's define $M = \mathrm{Im}(1_A)$, and $N = \mathrm{Im}(1_B)$. Since $(1_A)^2 = 1_A$, we see that 1_A is a projection onto the subspace $M \subset L$. Similarly 1_B is a projection onto N. Also, for any $x \in L$ we have

$$1_A(x) + 1_B(x) = 1_{A\oplus B}(x) = x$$

So $1_A(x) = x$ iff $1_B(x) = 0$, i.e. M is exactly the kernel of 1_B. Similarly N is exactly the kernel of 1_A. By Lemma 1.5.9 the vector space L splits as a direct sum

$$L = M \oplus N$$

and $1_A, 1_B$ are the projections onto the two summands.

We claim that both M and N are actually submodules of L. To prove this, suppose $x \in L$ lies in the the subspace M, *i.e.* suppose that $1_A(x) = x$. Then for any $(a, b) \in A \oplus B$ we have

$$\begin{aligned}(a, b)(x) &= (a, b)1_A(x) \\ &= (a, 0)(x) \\ &= 1_A(a, 0)(x)\end{aligned}$$

This lies in M, since M is the image of 1_A. This proves that M is a submodule. Futhermore the $A \oplus B$-module structure on M is really an A-module structure, since every element in B acts as zero on M.

By the same argument the subspace $N \subset A \oplus B$ is a submodule, and it's really a B-module. And we have already proven that $L = M \oplus N$. $\square$

Example 3.2.13. Let $A = \mathbb{C} \oplus \mathbb{C}$. Since a $\mathbb{C}$-module is nothing but a vector space (Example 3.2.3) every A-module is of the form $U \oplus W$ where U and W are vector spaces. Conversely, if U and W are any two vector spaces, then $U \oplus W$ is automatically an A-module. The element $(1, 0) \in A$ must act as the projection onto U, and the element $(0, 1) \in A$ acts as the projection onto W.

Notice that we can recover Example 3.2.6 immediately: a 1-dimensional A-module must have either $\dim U = 1$ and $\dim W = 0$, or vice-versa. In fact the proof of Proposition 3.2.12 is just a generalization of Example 3.2.6.

Now recall (Example 3.1.6) that we have an isomorphism of algebras

$$f : \mathbb{C}[C_2] \to A$$

sending:

$$e \mapsto (1, 1) \qquad \text{and} \qquad g \mapsto (1, -1)$$

This means that A-modules are the same thing as $\mathbb{C}[C_2]$-modules, *i.e.* representations of C_2. And since we understand modules over this algebra A completely, we can use this as an alternative way to classify the representations of C_2. Let's work this out explicitly.

We know that any A-module (*i.e.* any C_2-representation) splits as a direct sum $M \oplus N$ with $(1, 0)$ and $(0, 1)$ acting as the projection operators. Therefore $(1, -1)$ acts as the identity on M and minus the identity on N, *i.e.* our representation must send g to the block diagonal matrix:

$$g \mapsto \begin{pmatrix} \mathbf{1}_U & 0 \\ 0 & -\mathbf{1}_W \end{pmatrix}$$

So our representation splits as direct sum of copies of the trivial irrep U_1 and the sign irrep U_2, where the number of each is $\dim U$ and $\dim W$.

Of course the representation theory of the group C_2 is very easy! But for larger groups, you should start to believe that understanding the group algebra $\mathbb{C}[G]$ is closely related to understanding the representation theory of G.

3.3. Semi-simple modules and algebras of A-linear maps

Definition 3.3.1. Let A be an algebra. An A-module M is **simple** if it contains no (proper) submodules.

Example 3.3.2. Any 1-dimensional module must be simple.

Example 3.3.3. If $A = \mathbb{C}[G]$, then an A-module is simple if and only if the corresponding representation of G is irreducible. Really, 'simple' is just another word for 'irreducible'.

Example 3.3.4. Take a vector space V and set $A = \mathrm{Hom}(V, V)$. Then V is automatically an A-module (Example 3.2.5). We claim that V is simple.

If V was not simple we would be able to find a proper submodule, *i.e.* there would be a proper subspace

$$U \subset V$$

which is preserved by *every* linear map $f : V \to V$. But there is no such subspace.

Example 3.3.5. Take two vector spaces U and V and set:

$$A = \mathrm{Hom}(V, V) \oplus \mathrm{Hom}(U, U)$$

If we pick bases we can identity A with $\mathrm{Mat}_{d\times d}(\mathbb{C}) \oplus \mathrm{Mat}_{e\times e}(\mathbb{C})$, where $d = \dim V$ and $e = \dim U$. We could also identify A with a subalgebra of $\mathrm{Mat}_{(d+e)\times(d+e)}(\mathbb{C})$ as in Example 3.1.9.

In the previous example we observed that the algebra $\mathrm{Hom}(V, V)$ has a simple module V. We can also view V as a module over A, using the projection map

$$\mu : A \to \mathrm{Hom}(V, V)$$

(as we did in Proposition 3.2.12). Then V is still a simple A-module: if $W \subset V$ was a proper submodule then W would be preserved by every element of A, and in particular preserved by every $f \in \mathrm{Hom}(V, V)$. But since V is a simple $\mathrm{Hom}(V, V)$-module there is no such W. And by the same arguments, the vector space U is also a simple A-module.

So this algebra A has two obvious simple modules coming from the two factors. This generalizes Example 3.2.6 where we saw that the algebra $\mathbb{C} \oplus \mathbb{C}$ has two 1-dimensional modules coming from the two projection maps.

Schur's Lemma is really a fact about simple modules:

Proposition 3.3.6 (Schur's Lemma, v3)**.** *Let M and N be simple A-modules. Then*

$$\dim \mathrm{Hom}_A(M, N) = \begin{cases} 1 & \text{if } M \text{ and } N \text{ are isomorphic} \\ 0 & \text{if } M \text{ and } N \text{ are not isomorphic} \end{cases}$$

Proof. Use the identical proof to Theorem 1.6.1 and Proposition 1.7.1. □

Definition 3.3.7. An A-module M is **semi-simple** if it's isomorphic to a direct sum of simple A-modules.

This was not an important concept for representations of groups. This is because, thanks to Maschke's Theorem, every module over $\mathbb{C}[G]$ is semi-simple (because every representation is a direct sum of irreps). But for general algebras this is not true.

Example 3.3.8. Let $A = \langle 1, x \rangle$ with $x^2 = 0$. Let M be the A-module given by A itself. Let's look for 1-dimensional submodules of M, i.e. subspaces

$$\langle \lambda + \mu x \rangle \subset A$$

that are preserved under left-multiplication by all elements of A. We must have

$$x(\lambda + \mu x) = \lambda x \in \langle \lambda + \mu x \rangle$$

So $\lambda = 0$, and hence the subspace $\langle x \rangle$ is the only 1-dimensional submodule. So M is not simple (it contains a 1-dimensional submodule), but also it doesn't split up as a sum of a direct sum of simples, so M is not semi-simple either.

So another way that group algebras are special is that every $\mathbb{C}[G]$-module is semi-simple.

Our next task is to understand some algebras that appear when we consider A-linear maps. Fix an algebra A and an A-module M. The vector space

$$\mathrm{Hom}_A(M, M)$$

is an algebra, with the multiplication given by composition (the composition of two A-linear maps is clearly A-linear). It's a subalgebra of the larger algebra $\mathrm{Hom}(M, M)$ of all linear maps.

Example 3.3.9. Suppose M is simple. Then by Schur's Lemma we know:

$$\mathrm{Hom}_A(M, M) = \mathbb{C}$$

This is an isomorphism of algebras. Every A-linear map is of the form $\lambda \mathbf{1}_M$ for some scalar $\lambda \in \mathbb{C}$, and composing the linear maps corresponds to multiplying the scalars.

Example 3.3.10. Suppose M is a semi-simple module which is a direct sum $M = M_1 \oplus M_2$ of two non-isomorphic simple modules. Then again by Schur's Lemma we know

$$\mathrm{Hom}_A(M, M) = \mathrm{Hom}_A(M_1, M_1) \oplus \mathrm{Hom}_A(M_2, M_2) = \mathbb{C} \oplus \mathbb{C}$$

since there are no A-linear maps between M_1 and M_2 in either direction. This too is an isomorphism of algebras. An A-linear map from M to M must preserve both subspaces M_1 and M_2, and it must act as scalars on each subspace. So we can write it as

$$f = (\lambda \mathbf{1}_{M_1}, \mu \mathbf{1}_{M_2})$$

for some $\lambda, \mu \in \mathbb{C}$. Composing these maps agrees with the multiplication in $\mathbb{C} \oplus \mathbb{C}$.

Similarly if

$$M = M_1 \oplus .... \oplus M_r$$

is a direct sum of r simple modules, none of which are isomorphic, then $\mathrm{End}_A(M)$ is isomorphic to the algebra $\mathbb{C}^{\oplus r}$.

Example 3.3.11. Suppose N is a simple module and $M = N \oplus N$. Then Schur's Lemma tells us that $\dim \mathrm{Hom}_A(M, M) = 4$. We claim that as algebras we have:

$$\mathrm{Hom}_A(M, M) \cong \mathrm{Mat}_{2\times 2}(\mathbb{C})$$

Any linear map $f : N \oplus N \to N \oplus N$ is made of four components $f_{11}, f_{12}, f_{21}, f_{22}$, each of which is a linear map from N to N. For $(x, y) \in N \oplus N$ we must have:

$$f(x, y) = \big(f_{11}(x) + f_{12}(y),\ f_{21}(x) + f_{22}(y)\big)$$

If f is A-linear then each f_{ij} must be A-linear (Lemma 3.2.11), and since N is simple we must have $f_{ij} = \lambda_{ij}\mathbf{1}_N$ for some scalars $\lambda_{ij} \in \mathbb{C}$. So we can identify f with the 2×2 matrix whose entries are the λ_{ij}'s. And it's easy to check that composing A-linear maps corresponds to multiplying the matrices.

Notice that this has nothing to do with the dimension of N. If $\dim N = n$ then $\dim M = 2n$ and the algebra $\mathrm{Hom}(M, M)$ of all linear maps is isomorphic to $\mathrm{Mat}_{2n\times 2n}(\mathbb{C})$ (once we pick a basis for N). But $\mathrm{Hom}_A(M, M)$ is the 4-dimensional subalgebra consisting of all matrices of the form:

$$\begin{pmatrix} \lambda_{11}I_n & \lambda_{12}I_n \\ \lambda_{21}I_n & \lambda_{22}I_n \end{pmatrix}$$

If we understand the last few examples then the following generalization does not need a proof.

Lemma 3.3.12. *Let M be a semi-simple A-module of the form*

$$M = M_1^{\oplus k_1} \oplus .... \oplus M_r^{\oplus k_r}$$

where the M_i's are simple and no two are isomorphic. Then:

$$\mathrm{Hom}_A(M, M) \cong \mathrm{Mat}_{k_1\times k_1}(\mathbb{C}) \oplus ... \oplus \mathrm{Mat}_{k_r\times k_r}(\mathbb{C})$$

We are getting close to being able to prove the following theorem, which was promised at the end of Section 3.1.

Theorem 3.3.13. *Let G be a group, and let $d_1, ..., d_r$ be the dimensions of the irreps of G. Then we have an isomorphism of algebras:*

$$\mathbb{C}[G] \;\cong\; \mathrm{Mat}_{d_1\times d_1}(\mathbb{C}) \oplus ... \oplus \mathrm{Mat}_{d_r\times d_r}(\mathbb{C})$$

Notice that the dimensions of both sides agree by Corollary 1.7.11. But this doesn't mean automatically that they are the same algebra, we still have to prove that. And before we can give the proof we need one more definition.

Definition 3.3.14. Let A be an algebra. The **opposite algebra** A^{op} is the algebra with the same underlying vector space as A, and with multiplication

$$m^{op}(a, b) = ba$$

The algebra axioms for A imply immediately that A^{op} is an algebra, with the same unit $1_A \in A$. Obviously m^{op} is the same as the multiplication on A iff A is commutative. However, it's possible for a non-commutative algebra to still be *isomorphic* to its opposite, via an isomorphism which is not the identity map.

Example 3.3.15. (i) Let $A = \mathbb{C}[G]$ for a group G. Then the linear map defined by

$$A \to A^{op}$$
$$g \mapsto g^{-1}$$

is an isomorphism of algebras. It's clearly invertible, and it's an algebra homomorphism because $(gh)^{-1} = h^{-1}g^{-1}$.

(ii) Let $A = \mathrm{Mat}_{d\times d}(\mathbb{C})$. Then the linear map

$$\begin{aligned} A &\to A^{op} \\ M &\mapsto M^T \end{aligned}$$

is an isomorphism of algebras.

There do exist non-commutative algebras which are not isomorphic to their opposites, but it takes a bit of work to find them.

Now recall Lemma 1.7.14, which was a key ingredient for understanding irreps. It says that for any group G and any representation W we have an isomorphism of vector spaces:

$$\mathrm{Hom}_G(V_{reg}, W) \cong W$$

Let's translate this into our new language of algebras and modules. If we set $A = \mathbb{C}[G]$ then A-modules are the same thing as representations of G, and G-linear maps are the same thing as A-linear maps. Also the regular representation V_{reg} is the A-module given by A itself. So the obvious generalization is:

Lemma 3.3.16. *For any algebra A and any A-module M, there is an isomorphism of vector spaces*

$$\mathrm{Hom}_A(A, M) \cong M$$

Proof. This is exactly the same as the proof of Lemma 1.7.14. We use the map 'evaluate at 1_A':

$$\begin{aligned} T\colon\ \mathrm{Hom}_A(A, M) &\to M \\ f &\mapsto f(1_A) \end{aligned}$$

The inverse to T is the map

$$\begin{aligned} T^{-1} : M &\to \mathrm{Hom}_A(A, M) \\ x &\mapsto f_x \end{aligned}$$

where f_x is the A-linear map:

$$f_x : a \mapsto ax$$

$\square$

An important special case is when M is also the A-module given by A itself. In this case $\mathrm{Hom}_A(A, A)$ is not just a vector space, it's also an algebra.

Corollary 3.3.17. *For any algebra A there is an isomorphism of algebras:*

$$\mathrm{Hom}_A(A, A) \cong A^{op}$$

Proof. We know from Lemma 3.3.16 that for any A-module M we have an isomorphism of vector spaces

$$\begin{aligned} T^{-1}\colon A &\to \mathrm{Hom}_A(A, A) \\ b &\mapsto f_b \end{aligned}$$

where f_b is the A-linear map

$$f_b : a \mapsto ab$$

of 'right multiplication by b'. Then we just notice that

$$f_c \circ f_b : a \mapsto abc$$

so $f_c \circ f_b = f_{bc}$. So T^{-1} is an algebra homomorphism from A^{op} to $\mathrm{Hom}_A(A, A)$. $\square$

Now we can prove our theorem.

Proof of Theorem 3.3.13. Set $A = \mathbb{C}[G]$. Then the A-module A corresponds to the regular representation V_{reg}. We know from Theorem 1.7.8 that this A-module is isomorphic to a direct sum:

$$A \;\cong\; U_1^{\oplus d_1} \oplus \ldots \oplus U_r^{\oplus d_r}$$

Here the U_i are the irreps of G, *i.e.* the simple A-modules, and $d_i = \dim U_i$. By Corollary 3.3.17 and Lemma 3.3.12 we have

$$A^{op} \cong \mathrm{Hom}_A(A, A) \cong \mathrm{Mat}_{d_1 \times d_1}(\mathbb{C}) \oplus \ldots \oplus \mathrm{Mat}_{d_r \times d_r}(\mathbb{C})$$

and we also know that $A \cong A^{op}$ (Example 3.3.15(i)). □

Because we can write A as a direct sum of matrix algebras it has r 'obvious' simple modules given by projecting onto the factors (as in Example 3.3.5). These the modules corresponding to the irreps of G.

Theorem 3.3.13 doesn't just apply to group algebras.

Definition 3.3.18. An algebra A is called **semi-simple** if every A-module is semi-simple, *i.e.* if every A-module is a direct sum of simple modules.

Group algebras are semi-simple, but not all algebras are (Example 3.3.8). If A is a semi-simple algebra then in particular, the A-module given by A itself must be a semi-simple module. So we have an isomorphism of A-modules

$$A \cong U_1^{\oplus k_1} \oplus \ldots. \oplus U_r^{\oplus k_r}$$

for some simple modules $U_1, ..., U_r$. Then the proof of Theorem 3.3.13 tells us that we have an isomorphism of algebras:

$$A \;\cong\; \mathrm{Mat}_{k_1 \times k_1}(\mathbb{C}) \oplus \ldots \oplus \mathrm{Mat}_{k_r \times k_r}(\mathbb{C})$$

We have proved one direction of:

Theorem 3.3.19 (Artin-Wedderburn Theorem)**.** *An algebra A is semi-simple iff A is isomorphic to a direct sum of matrix algebras.*

Note that this is the version of the theorem that holds over $\mathbb{C}$. If you work over other fields the statement is more complicated.

To prove the other direction of the Artin-Wedderburn Theorem the key ingredient is:

Proposition 3.3.20. *Let V be a vector space and let A be the matrix algebra:*

$$A = \mathrm{Hom}(V, V)$$

Then every A-module is isomorphic to $V^{\oplus k}$ for some k.

We've already observed that the A-module V is simple (Example 3.3.4), so this Proposition tells us that A is a semi-simple algebra. And in only has one simple module, so it behaves a bit like a group that only has one irrep. The proof of the proposition is a bit fiddly so we have banished it to Appendix B.

Applying Proposition 3.2.12 we immediately deduce:

Corollary 3.3.21. *Let $V_1, ..., V_r$ be vector spaces and let A be the algebra*

$$A = \mathrm{Hom}(V_1, V_1) \oplus \ldots \oplus \mathrm{Hom}(V_r, V_r)$$

(so A is a direct sum of matrix algebras). Then every A-module is isomorphic to

$$V_1^{\oplus k_1} \oplus \ldots \oplus V_r^{k_r}$$

for some $k_1, ..., k_r$.

This says that A is a semi-simple algebra, with exactly r simple modules $V_1, .., V_r$. So we have now proved both directions of the Artin-Wedderburn Theorem.

The results we've proved in this section summarize a lot of our earlier discoveries about representations of groups. If G is a group then we've proved that $\mathbb{C}[G]$ is isomorphic to a direct sum of matrix algebras

$$\mathrm{Hom}(V_1, V_1) \oplus ... \oplus \mathrm{Hom}(V_r, V_r)$$

So we get r simple modules, *i.e.* irreps, from projecting onto the different factors in this direct sum. And we know that every module/representation is a direct sum of copies of these simple modules/irreps. In the case $G = C_2$ this is what we worked out explicity in Example 3.2.13.

3.4. Centres of algebras

Definition 3.4.1. Let A be an algebra. The **centre** of A is the subspace

$$Z_A = \{z \in A \mid za = az, \quad \forall a \in A\}$$

Z_A is a subalgebra of A, and it's commutative. Obviously $Z_A = A$ iff A is commutative.

Example 3.4.2. Let $A = \mathrm{Mat}_{d \times d}(\mathbb{C})$. Then it's easy to check that the centre of A is 1-dimensional:

$$Z_A = \{\lambda I_d \mid \lambda \in \mathbb{C}\}$$

These are the only matrices that commute with all other matrices. Abstractly, this says that if take a vector space V and set $A = \mathrm{Hom}(V, V)$ then $Z_A = \{\lambda \mathbf{1}_V \mid \lambda \in \mathbb{C}\}$.

Now let $A = \mathbb{C}[G]$ for a group G. If $g \in G$ is in the centre of G (*i.e.* if g commutes with all other group elements) then clearly g lies in Z_A, and in fact λg lies in Z_A for any $\lambda \in \mathbb{C}$. However, Z_A is usually larger than this.

Proposition 3.4.3. *Let $A = \mathbb{C}[G]$. Then $Z_A \subset A$ is spanned by the elements*

$$z_{[g]} = \sum_{h \in [g]} h \in A$$

for each conjugacy class $[g]$ in G.

Proof. Let $G = \{g_1, \ldots, g_k\}$, so a general element of $\mathbb{C}[G]$ looks like

$$a = \lambda_{g_1} g_1 + \ldots \lambda_{g_k} g_k$$

for some $\lambda_{g_1}, \ldots, \lambda_{g_k} \in \mathbb{C}$. Then a is in Z_A iff

$$ag = ga \quad \Longleftrightarrow \quad g^{-1} a g = a$$

for all $g \in G$. This holds iff

$$\lambda_{g g_i g^{-1}} = \lambda_{g_i}$$

for all g and g_i in G, *i.e.* iff

$$a = \sum_{\substack{\text{conjugacy classes} \\ [g] \text{ in } G}} \lambda_{[g]} z_{[g]}$$

for some scalars $\lambda_{[g]} \in \mathbb{C}$.

$\square$

The elements $z_{[g]}$ are obviously linearly independent, so $\dim Z_{\mathbb{C}[G]}$ is the number of conjugacy classes in G. In fact we can identify $Z_{\mathbb{C}[G]}$ with the space of class functions on G. If we identify $\mathbb{C}[G]$ with $\mathbb{C}^G$ by sending

$$g \leftrightarrow \delta_g$$

then the element $z_{[g]}$ maps to $\delta_{[g]}$, and $Z_{\mathbb{C}[G]}$ corresponds exactly to $\mathbb{C}^G_{cl} \subset \mathbb{C}^G$.

We are finally in a position to explain the proof of Theorem 2.3.3, that for any group G

$$\#\{\text{conjugacy classes in } G\} = \#\{\text{irreps of } G\}.$$

Proof of Theorem 2.3.3. Say G has r irreps, of dimensions $d_1, ..., d_r$. Then by Theorem 3.3.13 we have an isomorphism of algebras:

$$\mathbb{C}[G] \;\cong\; \mathrm{Mat}_{d_1\times d_1}(\mathbb{C}) \oplus ... \oplus \mathrm{Mat}_{d_r\times d_r}(\mathbb{C})$$

For any two algebras A, B it's obvious that $Z_{A\oplus B} = Z_A \oplus Z_B$, so

$$\begin{aligned} Z_{\mathbb{C}[G]} &\cong Z_{\mathrm{Mat}_{d_1\times d_1}(\mathbb{C})} \oplus ... \oplus Z_{\mathrm{Mat}_{d_r\times d_r}(\mathbb{C})} \\ &\cong \mathbb{C}^{\oplus r} \end{aligned}$$

(using Example 3.4.2). Hence $\dim Z_{\mathbb{C}[G]} = r$. But Proposition 3.4.3 tells us that $\dim Z_{\mathbb{C}[G]}$ is also the number of conjugacy classes in G. $\square$

Appendix A. Revision on linear maps and matrices

This appendix contains some brief revision material on the relationship between matrices and linear maps.

A.1. Vector spaces and bases

We let $\mathbb{C}^n$ be the set of column vectors

$$\begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_n \end{pmatrix}, \quad \lambda_i \in \mathbb{C}$$

This is a vector space of dimension n (over $\mathbb{C}$). It comes with its *standard basis* $e_1, e_2, ..., e_n$, where e_i is the column vector in which $\lambda_i = 1$ and all the other λ_j are zero. We can write the column vector above as $\sum_{i=1}^n \lambda_i e_i$.

Now let V be an abstract n-dimensional vector space (over $\mathbb{C}$).

Proposition A.1.1. *Choosing a basis for V is the same thing as choosing an isomorphism*

$$f : \mathbb{C}^n \xrightarrow{\sim} V$$

Proof. Suppose we've chosen such an isomorphism f. Then the images of the standard basis vectors under f give us a basis for V. Conversely, suppose $a_1, ..., a_n \in V$ is a basis. Define a linear map $f : \mathbb{C}^n \to V$ by mapping e_i to a_i, and then extending linearly. Then f is an isomorphism, because every vector $v \in V$ can be uniquely expressed as a linear combination

$$v = \lambda_1 a_1 + \lambda_2 a_2 + ... + \lambda_n a_n$$

$\square$

A.2. Linear maps and matrices

Proposition A.2.1. *Linear maps from $\mathbb{C}^n$ to $\mathbb{C}^m$ are the same thing as $m \times n$ matrices (with complex coefficients).*

Proof. Each matrix $M \in \mathrm{Mat}_{m\times n}(\mathbb{C})$ defines a linear map

$$\phi : \mathbb{C}^n \to \mathbb{C}^m$$

by multiplying column vectors by M (on the left). Notice that the image $\phi(e_i)$ of the ith standard basis vector is the column vector which forms the ith column of M.

Conversely, suppose $\phi : \mathbb{C}^n \to \mathbb{C}^m$ is any linear map. Let $M \in \mathrm{Mat}_{m\times n}(\mathbb{C})$ be the matrix whose columns are the vectors $\phi(e_1), \phi(e_2), ..., \phi(e_n)$, *i.e.*

$$\phi(e_i) = \sum_{j=1}^{m} M_{ji}\tilde{e}_j$$

where $\tilde{e}_1, ..., \tilde{e}_m$ is the standard basis for $\mathbb{C}^m$. Then for any column vector

$$v = \sum_{i=1}^{n} \lambda_i e_i \in \mathbb{C}^n$$

we have

$$\phi(v) = \sum_{i=1}^{n} \lambda_i \phi(e_i) \qquad = \sum_{i=1}^{n} \lambda_i \left(\sum_{j=1}^{m} M_{ji}\tilde{e}_j \right) \qquad = \sum_{j=1}^{m} \left(\sum_{i=1}^{n} M_{ji}\lambda_i \right) \tilde{e}_j$$

so the linear map ϕ is exactly multiplication by the matrix M. □

Corollary A.2.2. *Let V and W be abstract vector space of dimensions n and m respectively. Choose a basis $\mathcal{A} = \{a_1, ..., a_n\}$ for V, and a basis $\mathcal{B} = \{b_1, ..., b_m\}$ for W. Then we have a bijection between the set of linear maps from V to W and the set of $m \times n$ matrices.*

Proof. Because we've chosen bases for V and W, we have corresponding isomorphims

$$f_{\mathcal{A}} : \mathbb{C}^n \xrightarrow{\sim} V, \qquad\qquad f_{\mathcal{B}} : \mathbb{C}^m \xrightarrow{\sim} W$$

by Proposition A.1.1. Let

$$\varphi : V \to W$$

be a linear map. Then the composition

$$\phi = f_{\mathcal{B}}^{-1} \circ \varphi \circ f_{\mathcal{A}}$$

is a linear map from $\mathbb{C}^n$ to $\mathbb{C}^m$, so it has a corresponding matrix M. Conversely, any matrix $M \in \mathrm{Mat}_{m\times n}$ defines a linear map $\phi : \mathbb{C}^n \to \mathbb{C}^m$, and hence a linear map

$$\varphi = f_{\mathcal{B}} \circ \phi \circ f_{\mathcal{A}}^{-1} : V \to W$$

Clearly this gives a bijection between linear maps and matrices. □

Let's make this bijection between linear maps and matrices more explicit. Consider the diagram:

$$\begin{tikzcd} V \arrow[r, "\varphi"] & W \\ \mathbb{C}^n \arrow[u, "f_{\mathcal{A}}"] \arrow[r, "\phi"] & \mathbb{C}^m \arrow[u, "f_{\mathcal{B}}"'] \end{tikzcd}$$

By the definition of the matrix M, we have

$$\phi(e_i) = \sum_{j=1}^{m} M_{ji}\tilde{e}_j$$

Thus

$$\begin{aligned}
\varphi(a_i) &= \varphi(f_{\mathcal{A}}(e_i)) = f_{\mathcal{B}}(\phi(e_i)) \\
&= f_{\mathcal{B}}\left(\sum_{j=1}^{m} M_{ji}\tilde{e}_j\right) = \sum_{j=1}^{m} M_{ji} f_{\mathcal{B}}(\tilde{e}_j) \\
&= \sum_{j=1}^{m} M_{ji} b_j
\end{aligned}$$

So the ith column of M is the image of the basis vector a_i, expressed as a column vector using the basis $\mathcal{B}$.

Proposition A.2.3. *Let $\phi : \mathbb{C}^n \to \mathbb{C}^m$ be a linear map with corresponding matrix M, and let $\psi : \mathbb{C}^m \to \mathbb{C}^p$ be a linear map with corresponding matrix N. Then the composition $\psi \circ \phi$ corresponds to the matrix product NM.*

Proof. We have

$$\phi(e_i) = \sum_{j=1}^{m} M_{ji}\tilde{e}_j, \qquad \psi(\tilde{e}_j) = \sum_{k=1}^{p} N_{kj}\hat{e}_k$$

where $\hat{e}_1, ..., \hat{e}_p$ is the standard basis for $\mathbb{C}^p$. Therefore

$$\begin{aligned}
(\psi \circ \phi)(e_i) &= \psi\left(\sum_{j=1}^{m} M_{ji}\tilde{e}_j\right) && = \sum_{j=1}^{m} M_{ji}\psi(\tilde{e}_j) \\
&= \sum_{j=1}^{m} M_{ji}\left(\sum_{k=1}^{p} N_{kj}\hat{e}_k\right) && = \sum_{k=1}^{p}\left(\sum_{j=1}^{m} N_{kj}M_{ji}\right)\hat{e}_k \\
&= \sum_{k=1}^{p} (NM)_{ki}\hat{e}_k
\end{aligned}$$

So the matrix NM corresponds to $\psi \circ \phi$. ☐

A.3. Changing basis

Let V be an n-dimensional vector space, and let $\mathcal{A} = \{a_1, ..., a_n\} \subset V$ and $\mathcal{C} = \{c_1, ..., c_n\} \subset V$ be two different possible bases for V. Then by Proposition A.1.1 we have isomorphisms

$$\begin{tikzcd}
 & V & \\
\mathbb{C}^n \arrow[ur, "f_{\mathcal{A}}"] & & \mathbb{C}^n \arrow[ul, "f_{\mathcal{C}}"'] \arrow[ll, "f_{\mathcal{A}}^{-1}\circ f_{\mathcal{C}}"]
\end{tikzcd}$$

The composition $f_{\mathcal{A}}^{-1} \circ f_{\mathcal{C}}$ is an isomorphism from $\mathbb{C}^n$ to $\mathbb{C}^n$, so it correponds to some invertible $n \times n$ matrix P. We'll call P the *change-of-basis matrix* between $\mathcal{A}$ and $\mathcal{C}$.

Note that

$$c_i = f_{\mathcal{C}}(e_i) \qquad = f_{\mathcal{A}}(f_{\mathcal{A}}^{-1} \circ f_{\mathcal{C}}(e_i)) \qquad = f_{\mathcal{A}}\left(\sum_{j=1}^{n} P_{ji} e_j\right)$$
$$= \sum_{j=1}^{n} P_{ji} f_{\mathcal{A}}(e_j) \qquad = P_{1i}a_1 + P_{2i}a_2 + \dots + P_{ni}a_n$$

i.e. the entries of P tell you the coefficients of the vectors in the second basis $\mathcal{C}$, expressed using the first basis $\mathcal{A}$.

In particular, suppose that V is actually $\mathbb{C}^n$, and that $\mathcal{A} = \{e_1, \dots, e_n\}$ is the standard basis (so $f_{\mathcal{A}}$ is the identity map). Then the columns of P are the vectors that form the second basis $\mathcal{C}$. This gives a bijection between the set of possible bases for $\mathbb{C}^n$ and the set of invertible $n \times n$ matrices.

Now let V and W be two vector spaces, of dimensions n and m respectively. In Corollary A.2.2 we saw that linear maps from V and W corresponded to $n \times m$ matrices, *after* we'd chosen bases for V and W. For a given linear map φ, the matrix that we write down depends on our choice of bases, and if we change our choice of bases then the matrix that represents φ will also change.

Proposition A.3.1. *Let V and W be two vector spaces of dimensions n and m respectively, and let*

$$\varphi : V \to W$$

be a linear map. Let $\mathcal{A}$ be a basis for V and $\mathcal{B}$ be a basis for W, and let

$$M \in \mathrm{Mat}_{n\times n}(\mathbb{C})$$

be the matrix representing φ with respect to the bases $\mathcal{A}$ and $\mathcal{B}$. Now let $\mathcal{C}$ be a second choice of basis for V, and let N be the matrix representing φ with respect to the bases $\mathcal{C}$ and $\mathcal{B}$. Then

$$N = MP$$

where P is the change-of-basis matrix between $\mathcal{A}$ and $\mathcal{C}$.

Proof. Examine the following diagram:

$$\begin{tikzcd}
\mathbb{C}^n \arrow[dr, "f_{\mathcal{C}}"] \arrow[dd, "f_{\mathcal{A}}^{-1}\circ f_{\mathcal{C}}"'] \arrow[rrrd, bend left, "f_{\mathcal{B}}^{-1}\circ\varphi\circ f_{\mathcal{C}}"] & & & \\
& V \arrow[r, "\varphi"] & W & \mathbb{C}^m \arrow[l, "f_{\mathcal{B}}"'] \\
\mathbb{C}^n \arrow[ur, "f_{\mathcal{A}}"'] \arrow[rrru, bend right, "f_{\mathcal{B}}^{-1}\circ\varphi\circ f_{\mathcal{A}}"'] & & &
\end{tikzcd}$$

We have

$$(f_{\mathcal{B}}^{-1} \circ \varphi \circ f_{\mathcal{C}}) = (f_{\mathcal{B}}^{-1} \circ \varphi \circ f_{\mathcal{A}}) \circ (f_{\mathcal{A}}^{-1} \circ f_{\mathcal{C}})$$

These three linear maps correspond to the matrices N, M and P respectively, so

$$N = MP$$

by Proposition A.2.3. □

Now suppose that $\mathcal{D}$ is a second choice of basis for W, and let Q be the change-of-basis matrix between $\mathcal{B}$ and $\mathcal{D}$. A very similar proof shows that the matrix representing φ with respect to the bases $\mathcal{A}$ and $\mathcal{D}$ is

$$Q^{-1}M$$

and that the matrix representing φ with respect to $\mathcal{C}$ and $\mathcal{D}$ is

$$Q^{-1}MP$$

Now let's specialize to the case that W and V are actually the same vector space:

Corollary A.3.2. *Let V be an n-dimensional vector space.*

i) If we choose a basis $\mathcal{A}$ for V, then we get a bijection between $\mathrm{Mat}_{n\times n}(\mathbb{C})$ and the set of linear maps from V to V.

ii) Let

$$\varphi : V \to V$$

be a linear map, and let M be the matrix representing φ with respect to the basis $\mathcal{A}$. Now let $\mathcal{C}$ be another basis for V, and let P be the change-of-basis matrix between $\mathcal{A}$ and $\mathcal{C}$. Then the matrix representing φ with respect to the basis $\mathcal{C}$ is

$$P^{-1}MP$$

Appendix B. Modules over matrix algebras

Let V be a vector space. In this section we'll study modules over the matrix algebra:

$$A = \mathrm{Hom}(V, V)$$

We have already observed that A has one obvious module given by V, *i.e.* by letting

$$\mu : A \to \mathrm{Hom}(V, V)$$

be the identity. We have also observed that V is a simple A-module (Example 3.3.4). Our goal is to prove Proposition 3.3.20 which says that every A-module is just a direct sum of copies of V, so A is a semi-simple algebra with a unique simple module.

First let's deal with the other obvious A-module, which is A itself.

Lemma B.0.1. *Let V be a vector space of dimension n and let $A = \mathrm{Hom}(V, V)$. Then A is isomorphic as an A-module to $V^{\oplus n}$.*

Proof. Pick a basis for V so A becomes $\mathrm{Mat}_{n\times n}(\mathbb{C})$. Then A acts on itself by matrix multiplication, and it acts on $V \cong \mathbb{C}^n$ via the action of matrices on column vectors. Consider the subspace

$$A_{\bullet k} \subset A$$

of matrices which are zero except in the kth column, *i.e.* they look like

$$\begin{pmatrix} 0 & \dots & 0 & a_1 & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & a_n & 0 & \dots & 0 \end{pmatrix}$$

Then each $A_{\bullet k}$ is a obviously a submodule, and it's obviously isomorphic to V. Also

$$A = \bigoplus_{k=1}^{n} A_{\bullet k}$$

□

Corollary B.0.2. *For any A-module M, we have*

$$\dim M = nk$$

where $k = \dim \mathrm{Hom}_A(V, M)$.

Proof. By Lemma 3.3.16 and Lemma B.0.1 we have isomorphisms of vector spaces:

$$M \cong \mathrm{Hom}_A(A, M) \cong \mathrm{Hom}_A(V^{\oplus n}, M) \cong \mathrm{Hom}_A(V, M)^{\oplus n}$$

$\square$

This already puts a sharp restriction on possible A-modules, they can only have dimensions which are a multiple of $n = \dim V$. And notice that if you know your module M is of the form $V^{\oplus k}$ then we must have $k = \dim \mathrm{Hom}_A(V, M)$ (that's basically Proposition 1.7.5). But proving that M must be of this form is the hard part.

In fact it's one of the most difficult results in this course. We're going to give two proofs, the first one is more elegant, the second one is quicker and messier.

First proof of Proposition 3.3.20. Let M be an A-module and let $k = \dim \mathrm{Hom}_A(V, M)$. Corollary B.0.2 says that M and $V^{\oplus k}$ have the same dimension and hence are isomorphic as vector spaces. But we need to prove that they're isomorphic as A-modules.

Let $\{f_1, ..., f_k\}$ be a basis for $\mathrm{Hom}_A(V, M)$, and let:

$$F = (f_1, \ldots, f_k) \in \mathrm{Hom}_A(V, M)^{\oplus k} = \mathrm{Hom}_A(V^{\oplus k}, M)$$

So F is an A-linear map from $V^{\oplus k}$ to M. We want to prove that F is an isomorphism, and in fact it's sufficient to prove that it's an injection because both modules have the same dimension.

Consider the maps

$$F_{\leq l} = (f_1, ..., f_l) \colon V^{\oplus l} \to M$$

where $1 \leq l \leq k$. We're going to prove by induction that each map $F_{\leq l}$ is an injection. Since $F = F_{\leq k}$ this will prove the proposition.

To begin the induction we first need to show that $F_{\leq 1} = f_1 : V \to M$ is an injection. The kernel of f_1 is a submodule of V, but V is a simple module, and f_1 is not the zero map, so we must have $\mathrm{Ker}(f_1) = \{0\}$. So $F_{\leq 1}$ is indeed an injection. Note that the same argument shows that each map $f_i : V \to M$ must be an injection.

Now take $1 \leq l < k$, and assume (for the inductive step) that

$$F_{\leq l} : V^{\oplus l} \to M$$

is an injection. Let

$$N_{\leq l} \subset M$$

be the image of $F_{\leq l}$. Then $N_{\leq l}$ is a submodule of M which is isomorphic to $V^{\oplus l}$ (because $F_{\leq l}$ is an injection). Also let

$$N_{l+1} \subset M$$

be the image of the map f_{l+1}. This is a submodule which is isomorphic to V. Now we consider the intersection:

$$N_{\leq l} \cap N_{l+1}$$

The intersection of any two submodules is a submodule (this follows immediately from the definition of a submodule), so $N_{\leq l} \cap N_{l+1}$ is a submodule of N_{l+1}. But $N_{l+1} \cong V$ is a simple A-module so we must have either:

$$N_{\leq l} \cap N_{l+1} = N_{l+1} \qquad \text{or} \qquad N_{\leq l} \cap N_{l+1} = \{0\}$$

Let us suppose, for a contradiction, that the first one is true, *i.e.* that N_{l+1} is contained in $N_{\leq l}$. This implies that f_{l+1} defines an A-linear map:

$$f_{l+1} : V \to N_{\leq l}$$

Each of the maps $f_1, ..., f_l$ also defines an A-linear map from V to $F_{\leq l}$, so we have a subset

$$\{f_1, ..., f_{l+1}\} \subset \mathrm{Hom}_A(V, N_{\leq l})$$

of size $l + 1$. However, the dimension of this space is:

$$\begin{aligned} \dim \mathrm{Hom}_A(V, N_{\leq l}) &= \dim \mathrm{Hom}_A(V, V^{\oplus l}) \quad \text{(by the inductive hypothesis)} \\ &= \dim \mathrm{Hom}_A(V, V)^{\oplus l} \; = \; l \end{aligned}$$

Therefore the maps $f_1, ..., f_{l+1}$ cannot be linearly independent, which contradicts the fact that $\{f_1, ..., f_k\}$ is a basis for $\mathrm{Hom}_A(V, M)$. We conclude that in fact N_{l+1} cannot be containined in $N_{\leq l}$, and therefore:

$$N_{\leq l} \cap N_{l+1} = \{0\}$$

The image of the map

$$F_{\leq l+1} : V^{\oplus l+1} \to M$$

is a submodule of M, and it's spanned by the two submodules $N_{\leq l}$ and N_{l+1}. Since these have trivial intersection, it follows that

$$\mathrm{Im}(F_{\leq l+1}) \cong N_{\leq l} \oplus N_{l+1} \cong V^{\oplus l} \oplus V$$

So $F_{\leq l+1}$ is an isomorphism onto its image, *i.e.* it's an injection. This proves the inductive step and completes the proof of the proposition.

□

Second proof of Proposition 3.3.20. We use the same A-linear map

$$F : V^{\oplus k} \to M$$

as in the first proof above. This time, we're going to prove that F is a surjection.

Pick any $x \in M$. We need to show that x is in the image of F. Under the isomorphism between M and $\mathrm{Hom}_A(A, M)$, the vector x corresponds to the A-linear map

$$\begin{aligned} T_x \colon A &\to M \\ a &\mapsto ax \end{aligned}$$

In particular, $T_x(1_A) = x$. Now pick a basis for V, so we can identify V with $\mathbb{C}^n$ (the space of column vectors). We can also identify A with $\mathrm{Mat}_{n\times n}(\mathbb{C})$, and we have A-linear injections

$$\iota_t \colon V \to A$$

given by mapping $\mathbb{C}^n$ to the space $A_{\bullet t}$ of matrices which are zero outside the tth column (as in Lemma B.0.1). Let $e_1, ..., e_n$ be the standard basis for $\mathbb{C}^n$, then:

$$\sum_{t=1}^{n} \iota_t(e_t) = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix} = I_n = 1_A$$

So:

$$T_x\left(\sum_{t=1}^{n} \iota_t(e_t)\right) = \sum_{t=1}^{n} (T_x \circ \iota_t)(e_t) = x \quad \in M$$

Now each map $T_x \circ \iota_t$ is in $\mathrm{Hom}_A(V, M)$, so it can be written as a linear combination of the basis vectors
$$T_x \circ \iota_t = \lambda_t^1 f_1 + \ldots + \lambda_t^k f_k$$
for some coefficients $\lambda_t^i \in \mathbb{C}$. Therefore
$$\begin{aligned} x = \sum_{t=1}^n (T_x \circ \iota_t)(e_t) &= \sum_{t=1}^n \sum_{i=1}^k \lambda_t^i f_i(e_t) \\ = \sum_{i=1}^k f_i \left(\sum_{t=1}^n \lambda_t^i e_t \right) &= F\left(\sum_{t=1}^n \lambda_t^1 e_t, \ldots, \sum_{t=1}^n \lambda_t^k e_t \right) \end{aligned}$$
So F is indeed surjective. $\square$

Aside: What we've actually proved is that M is isomorphic as an A-module to
$$V \otimes \mathrm{Hom}_A(V, M)$$
where we give the latter an A-module structure by letting A act only on the V factor.